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Some Properties of Pseudo-Differential Operators Involving Fractional Fourier Transform

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Abstract

Some properties of pseudo-differential operators on Schwartz space $S(\mathrm{R}^n)$ are studied by using the fractional Fourier transform.

1. INTRODUCTION

The concept of pseudo-differential operators was originated by Kohn-Nirenberg [1], Hörmander [2, 3] and others. Pseudo-differential operators on $S(\mathbb{R})$ have been discussed in association with fractional Fourier transform by Pathak et al. [4], Prasad and Kumar [5]. In this connection, pseudo-differential operators of infinite order involving fractional Fourier transform on $W_M(\mathbb{R}^n)$ and $W_M^{\Omega}(\mathbb{C}^n)$ have been studied by Upadhyay et al. [6], Upadhyay and Dubey [7] respectively. The main aim of this paper is to discuss the some properties of pseudo-differential operators involving fractional Fourier transform on Schwartz space $S(\mathbb{R}^n)$, where $\mathbb{R}^n$ is usual Euclidean space. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$are the elements of $\mathbb{R}^n$.

Then the inner product of *x* and *y* is defined by

$$\langle x, y\rangle = x.y = \sum_{j=1}^{n} x_j \,.\, y_j \tag{1.1}$$

and the norm of *x* is defined by

$$|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} \tag{1.2}$$

If $\beta = (\beta_1, \dots, \beta_n)$ is an n-tuple of non-negative integers, then β is called a multi-indices. We write, $|\beta| = \beta_1 + \cdots + \beta_n$ and for $x \in \mathbb{R}^n, x^\beta = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$. The *n*-dimensional fractional Fourier transform with parameter α of $\phi(x)$ on $x \in \mathbb{R}^n$ is denoted by $(F_\alpha \phi)(\xi) = \hat{\phi}_\alpha(\xi)$ [8, 6] and defined as

$$\hat{\phi}_\alpha(\xi) = (F_\alpha \phi)(\xi) = \int_{\mathbb{R}^n} K_\alpha(x, \xi)\phi(x)dx, \xi \in \mathbb{R}^n \tag{1.3}$$

where

$$K_\alpha(x,\xi) = \begin{cases} C_\alpha e^{\frac{i(|x|^2+|\xi|^2)\cot\alpha}{2} - i\langle x,\xi\rangle \csc\alpha} & if\ \alpha \neq n\pi \\ \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-i\langle x,\xi\rangle} & if\ \alpha = \frac{\pi}{2}, \end{cases} \quad \forall \mathrm{n} \in \mathbb{Z},$$

and

$$C_\alpha = (2\pi i \sin\alpha)^{-\frac{n}{2}} e^{\frac{in\alpha}{2}}$$

The corresponding inversion formula is given by

$$\phi(x) = \int_{\mathbb{R}^n} \overline{K_\alpha(x,\xi)}\, \hat{\phi}_\alpha(\xi) d\xi, x \in \mathbb{R}^n \tag{1.4}$$

where the kernel

$$\overline{K_\alpha(x,\xi)} = C'_\alpha e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2}+i\langle x,\xi\rangle \csc\alpha},$$

and

$$C'_\alpha = \frac{(2\pi i \sin\alpha)^{\frac{n}{2}}}{(2\pi \sin\alpha)^n} e^{\frac{-in\alpha}{2}}. \tag{1.5}$$

Definition 1.1 The Schwartz space $S(\mathbb{R}^n)$ is the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\gamma_{\mu,v}(\phi) = \sup_{x\,\in\,\mathbb{R}^n} |x^\mu D^v \phi(x)| < \infty, \tag{1.6}$$

for all multi-indices μ and v.

Definition 1.2 Let $m \in \mathbb{R}$. Then we define the symbol class S^m to be the space of all $\theta(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any two multi-indices μ and v, there is a positive constant $C_{\mu,v}$ depending upon μ and v such that

$$\left|\left(D_x^\mu D_\xi^v\right)(x,\xi)\right| \leq C_{\mu,v}(1+|\xi|)^{m-|v|}, \qquad \forall\, x,\xi \in \mathbb{R}^n. \tag{1.7}$$

Definition 1.3 Let $\theta(x,\xi)$ be a symbol belonging to S^m, then the pseudodifferential operator $A_{\theta,\alpha}$ associated with $\theta(x,\xi)$ is defined as

$$(A_{\theta,\alpha}\phi)(x) = \int_{\mathbb{R}^n} \overline{K_\alpha(x,\xi)}\, \theta(x,\xi)\hat{\phi}_\alpha(\xi)d\xi, \phi \in S(\mathbb{R}^n) \tag{1.8}$$

where $\hat{\phi}_\alpha(\xi)$ is the fractional Fourier transform of $\phi(x)$, defined by (1.3), and

$$\overline{K_\alpha(x,\xi)} = C'_\alpha e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2}+i\langle x,\xi\rangle \csc\alpha}, \tag{1.9}$$

where C'_α is defined by (1.5).

2. PSEUDO-DIFFERENTIAL OPERATORS INVOLVING FRACTIONAL FOURIER TRANSFORM

In this section, some properties of pseudo-differential operator associated with fractional Fourier transform defined by (1.8) on $S(\mathbb{R}^n)$ are discussed.

Theorem 2.1 Let $\theta(x,\xi) \in S^m$, where $m \in \mathbb{R}$. Then $A_{\theta,\alpha}$ maps $S(\mathbb{R}^n)$ into itself.

Proof. Let $\phi \in S(\mathbb{R}^n)$. Then for any two multi-indices μ and v, we need only prove that

$$\sup_{x\,\in\,\mathbb{R}^n} \left|x^\mu D_x^v (A_{\theta,\alpha}\phi)(x)\right| < \infty. \tag{2.1}$$

Proof of the above theorem follows from the facts of Pathak et. al. [4, Theorem 4.1] and Wong [9, pp. 31-32].

Theorem 2.2 Let $\theta(\xi) \in C^k(\mathbb{R}^n), k > \frac{n}{2}$, be such that there is a positive constant $C_{\beta,n}$ depending on β and n only, such that

$$\left|\left(D_\xi^\beta \theta\right)(\xi)\right| \leq C_{\beta,n}(1+|\xi|)^{-\beta} \tag{2.2}$$

for multi-indices β with $|\beta| \leq k$. Then for $1 \leq p < \infty$, there exists a positive constant $B'_{\alpha,\beta,n}$ depending on α,β and n only, we have

$$\left\|(A_{\theta,\alpha}\phi)(x)\right\|_p \leq B'_{\alpha,\beta,n} \|\phi(x)\|_p \;\forall \phi(x) \in S(\mathbb{R}^n), \tag{2.3}$$

where

$$(A_{\theta,\alpha}\phi)(x) = C'_\alpha \int_{\mathbb{R}^n} e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2}+i\langle x,\xi\rangle \csc\alpha}\, \theta(\xi)\hat{\phi}_\alpha(\xi)d\xi. \tag{2.4}$$

Proof. From (2.4), we can write

$$(A_{\theta,\alpha}\phi)(x) = F_\alpha^{-1}\left[\theta(\xi)\hat{\phi}_\alpha(\xi)\right](x). \tag{2.5}$$

Now we assume that

$$F_\alpha^{-1}\big[\theta(\xi)\hat{\phi}_\alpha(\xi)\big](x) = (f * g)(x).$$

Then,

$$\theta(\xi)\hat{\phi}_\alpha(\xi) = F_\alpha[(f*g)(x)](\xi),$$
$$= C_\alpha \int_{\mathbb{R}^n} e^{\frac{i(|x|^2+|\xi|^2)\cot\alpha}{2} - i\langle x,\xi\rangle \csc\alpha}(f*g)(x)dx.$$

From the arguments of [7, pp. 121, 122], we obtain

$$\theta(\xi)\hat{\phi}_\alpha(\xi) = \frac{1}{C_\alpha} \times e^{\frac{-i|\xi|^2 \cot x}{2}} \hat{f}_\alpha(\xi)\hat{g}_\alpha(\xi). \tag{2.6}$$

From (2.6), we get

$$\theta(\xi) = \frac{1}{C_\alpha} \times e^{\frac{-i|\xi|^2 \cot\alpha}{2}} \hat{f}_\alpha(\xi), \;\; \hat{\phi}_\alpha(\xi) = \hat{g}_\alpha(\xi). \tag{2.7}$$

Therefore,

$$f(x) = C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right](x), \;\; g(x) = \phi(x). \tag{2.8}$$

Thus, the (2.5) gives

$$\left(A_{\theta,\alpha}\phi\right)(x) = \left(C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right] * \phi\right)(x) \tag{2.9}$$

Using convolution property $\|f*\phi\|_p \le \|f\|_1\|\phi\|_p$ for $f \in L^1(\mathbb{R}^n)$ and $\phi \in L_p(\mathbb{R}^n)$, we have

$$\left\|\left(A_{\theta,\alpha}\phi\right)(x)\right\|_p = \left\|\left(C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right] * \phi\right)(x)\right\|_p$$
$$\le \left\|C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right](x)\right\|_1 \|\phi(x)\|_p. \tag{2.10}$$

Now, we shall prove that

$$C_\alpha F_\alpha^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x) \in L^1(\mathbb{R}^n). \tag{2.11}$$

Thus,

$$F_\alpha^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x) = \frac{C'_\alpha}{C_\alpha}\int_{\mathbb{R}^n} e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2} + i\langle x,\xi\rangle\csc\alpha} e^{\frac{i|\xi|^2\cot x}{2}}\theta(\xi)\,d\xi$$
$$= \frac{C'_\alpha}{C_\alpha}\int_{\mathbb{R}^n} e^{\frac{-i|x|^2\cot\alpha}{2} + i\langle x,\xi\rangle\csc\alpha}\theta(\xi)\,d\xi.$$

Then

$$x^\beta F_\alpha^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x) = \frac{C'_\alpha}{C_\alpha}(i\csc\alpha)^{-|\beta|} e^{\frac{-i|x|^2\cot\alpha}{2}}\int_{\mathbb{R}^n} D_\xi^\beta\left(e^{i\langle x,\xi\rangle\csc\alpha}\right)\theta(\xi)\,d\xi.$$

By using the parts of integration, we get

$$x^\beta F_\alpha^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x) = \frac{C'_\alpha}{C_\alpha}(-1)^{|\beta|}(i\csc\alpha)^{-|\beta|} e^{\frac{-i|x|^2\cot\alpha}{2}}\int_{\mathbb{R}^n}\left(e^{i\langle x,\xi\rangle\csc\alpha}\right)(D_\xi^\beta\theta)(\xi)\,d\xi.$$

Therefore,

$$\left|x^\beta F_\alpha^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x)\right| \le \left|\frac{C'_\alpha}{C_\alpha}(\csc\alpha)^{-|\beta|}\right|\int_{\mathbb{R}^n}\left|\left(e^{i\langle x,\xi\rangle\csc\alpha}\right)\right|\left|(D_\xi^\beta\theta)(\xi)\right|d\xi.$$

Using (2.2), we have

$$\left|x^{\beta}F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x)\right| \le \left|\frac{C'_{\alpha}}{C_{\alpha}}(\csc\alpha)^{-|\beta|}\right| C_{\beta,n}\int_{\mathbb{R}^n}(1+|\xi|)^{-|\beta|}\,d\xi.$$

Thus the last integral is convergent for sufficiently large β, then we get

$$\left|x^{\beta}F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x)\right| \le \left|\frac{C'_{\alpha}}{C_{\alpha}}(\csc\alpha)^{-|\beta|}\right| C_{\beta,n}C_{\beta}.$$

Hence, we have

$$\left|F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x)\right| \le C_{\beta,n,\csc\alpha}(1+|x|)^{-|\beta|}.$$

Then,

$$\begin{aligned}\left\|F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x)\right\|_1 &\le C_{\beta,n,\csc\alpha}\left\|(1+|x|)^{-|\beta|}\right\|_1\\ &< \infty.\end{aligned}$$

Therefore,

$$F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^2\cot\alpha}{2}}\theta(\xi)\right)(x)\in L^1(\mathbb{R}^n).$$

From (2.10), we get

$$\begin{aligned}\left\|(A_{\theta,\alpha}\phi)(x)\right\|_p &\le C_{\beta,n,\csc\alpha}\left\|(1+|x|)^{-|\beta|}\right\|_1\|\phi(x)\|_p\\ &\le B'_{\alpha,\beta,n}\|\phi(x)\|_p, \text{ for } (x)\in S(\mathbb{R}^n).\end{aligned}$$

A similar theorem has been studied by Upadhyay et al. [10] on $W_M(\mathbb{R}^n)$ space by using Fourier transform.

Example 2.1 Consider a generalized differential operator, which is defined by $\triangle_x^s=\triangle_{x_1}^{s_1}\cdots\triangle_{x_n}^{s_n}$, where $s\in\mathbb{Z}_+^n$, $x\in\mathbb{R}^n$ and for each j = 1, 2, · · · n, we have

$$\triangle_{x_j}^{s_j}=\left(-i\frac{\partial}{\partial x_j}+x_j\cot\alpha\right)^{s_j}.$$

Let $\phi\in S\,(\mathbb{R}^n)$, then using (1.4) we have

$$\begin{aligned}\triangle_x^s\,\phi(x) &=\triangle_x^s\int_{\mathbb{R}^n}\overline{K_\alpha(x,\xi)}\,\hat{\phi}_\alpha(\xi)d\xi, \qquad \forall\xi\in\mathbb{R}^n\\ &=\left(-i\frac{\partial}{\partial x}+x\cot\alpha\right)^s C'_\alpha\int_{\mathbb{R}^n}e^{\frac{-i(|x|^2+|\xi|^2)\cot x}{2}+i\langle x,\xi\rangle\csc\alpha}\hat{\phi}_\alpha(\xi)d\xi\\ &=\int_{\mathbb{R}^n}\overline{K_\alpha(x,\xi)}\,(\xi\csc\alpha)^s\hat{\phi}_\alpha(\xi)d\xi.\end{aligned}$$

Since $(\xi\ \csc\alpha)^s\in S^m$ for $m=|s|=s_1+\cdots s_n$ where $m\in\mathbb{Z}_+$. Hence generalised differential operator $\triangle_x^s$ is a pseudo-differential operator with symbol $(\xi\ \csc\alpha)^s$ in the sense of fractional Fourier transform.

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