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Fractional Order Neutral Mixed Integrodifferential Evolution Equations

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Abstract

Existence of a mild solution of the neutral integrodifferential equations with fractional order is established. We used the methods of the fractional power of operators as well as the Sadovskii's fixed point theorem for getting the main outcomes.

1. INTRODUCTION

Differential equations of fractional order have appeared in many applications in a lot of realm of engineering and science see, for instances, [1-5]. For more information on th is idea and applications, we go into the monographs of Lakshmikantham et al. [6], Miler and Ross [7], Podlubny [8] and Kilbas and Srivastava [9].

Instead of the above, Neutral differential equations are occurred in issues have to do with electric networks incorporating lossless transmission lines. Such types of networks emerge, for instance, in high speed computers in which lossless transmission lines are employed to interconnect switching circuits. Because of these types of equations have gained more attention in last few decades; see, for example, Hern´andez et. al. [10], Hern´andez [11], Valliammal et. al. [12], Chandrasekaran et.al. [13], Fu et. al. [14], Manimaran et.al.[15], and references cited therein. A good vanguard to the literature for neutral functional differential equations, we mention the book [16].

Besides of the above, integrodifferential equations evolve in variety of realm like electromagnetics, biological models, optics, inverse scattering problems and other practical applications. The models of basic electric circuit analysis are very fine illustrations of these equations. Fractional integrodifferential equation evolves in variety of realm of engineering like heat conduction of materials with memory and optimal control problem etc. Neutral integrodifferential equations with fractional order have been investigated a lot of researchers [17-23]. In this paper we generalize the results of [19, 24, 25]

In the present work, we have assumed the neutral fractional integrodifferential equations of the following form

$$
\begin{aligned}
&{}^{c}D^{\beta}\left[z(\iota)+F_1\left(\iota,z(\iota),z(\psi_1(\iota)),\dots,z(\psi_m(\iota))\right)\right]+Az(\iota)=F_2\left(\iota,z(\iota),z(d_1(\iota)),\dots,z(d_n(\iota))\right)\\
&\quad +F_3\left(\iota,z(\iota),\int_0^{\iota}k_1(\iota,s,z(s))\,ds,\int_0^{b}k_2(\iota,s,z(s))\,ds\right),\iota\in I=[0,b],\\
&\quad z(0)+k(z)=z_0, \qquad \text{...(1)}
\end{aligned}
$$

where $-A$ generates an analytic semigroup, and F_1,F_2,F_3,k_1,k_2 and k are functions which is mentioned in equation (1) and is defined later. The fractional derivative ${}^{c}D^{\beta}, 0<\beta<1$ is understood in the Caputo sense.

Throughout the paper, a Banach space E together the norm $\|\cdot\|$. Assume that $-A$ is a infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $S(\iota)$. This implies that $\exists\ M_1\geq 1$ in such a way that $\|S(\iota)\|\leq M_1$. We assume that $0\in\rho(A)$, then represent the fractional power $A^{\delta},\ 0<\delta\leq 1,$ like closed linear operator within its domain $D(A^{\delta})$ with inverse $A^{-\delta}$ containing the fundamental characteristic mentioned below.

Theorem 1.1 (see [26])

(i) $E_{\delta}=D(A^{\delta})$ is a Banach space with the norm $\|z\|_{\delta}=\|A^{\delta}z\|, z\in E_{\delta}$.

(ii) $S(\iota):E\to E_{\delta},\ \forall\ \iota>0$ and $A^{\delta}S(\iota)z=S(\iota)A^{\delta}z\ ,\forall z\in E_{\delta}$ and $\iota\geq 0$.

(iii) For all $\iota > 0, A^{\delta}S(\iota)$ is bounded on E and we have a positive constants C_{δ} such that

$$\left\| A^{\delta}S(\iota)\right\| \le \frac{C_{\delta}}{\iota^{\delta}}. \qquad ...(2)$$

(iv) If $0 < \alpha < \delta \le 1,$ then $D(A^{\delta}) \hookrightarrow D(A^{\alpha})$ and embedding is compact if the resolvent operator of A is compact.

Furthermore, we memories the well known definitions noted below.

Definition 1.1 (see [7, 8]). The fractional integral of order $\beta > 0$ with the lower limit zero for a function g can be termed as

$$I^{\beta} g\ (\iota) = \frac{1}{\Gamma\beta}\int_0^{\iota} \frac{\mathrm{g(s)}}{(\iota - s)^{1-\beta}} \mathrm{ds}, \iota > 0$$

Provided the right-hand side is point wise given on $[0,\infty)$ where $\Gamma(.)$ is Gamma function.

Definition 1.2 (see [7, 8]). The Caputo derivative of order β with the lower limit zero for a function g can be given as

$${}^{c}D^{\beta} g(\iota) = \frac{1}{\Gamma(n-\beta)}\int_0^{\iota} \frac{g^{(n)}(s)}{(\iota - s)^{\beta+1-n}} ds = I^{n-\beta} f^{(n)}(\iota), \iota > 0, 0 \le n-1 < \beta < n.$$

The integrals that looking in the above mentioned definitions are extracting in Bochner's sense whenever g will be an abstract function with values in E.

Now, the following conditions are stated:

(A_1) The continuous function $F_1 : [0,b]\times E^{m+1} \to E$, and $\exists$ a constant $\alpha \in (0,1)$ and $M_2, M_3 > 0$ intending to $A^{\alpha}F_1$ meets the Lipschitz condition:

$$\left\| A^{\alpha}F_1(s_1, z_0, z_1, \ldots, z_m) - A^{\alpha}F_1(s_2, \overline{z_0}, \overline{z_1}, \ldots, \overline{z_m})\right\| \le M_2\left(|s_1 - s_2| + \max_{i=0,\ldots,m}\left\| z_i - \overline{z_i}\right\|\right),$$

for any $0 \le s_1, s_2 \le b, z_i, \overline{z_i} \in E, i = 0,1,\ldots,m;$ and the inequality is given as

$$\left\| A^{\alpha}F_1(\iota, z_0, z_1, \ldots, z_m)\right\| \le M_3\left(\max_{i=0,\ldots,m}\{\|z_i\| : i = 0,1,\ldots,m\} + 1\right), \qquad ...(3)$$

holds $\forall\ (\iota, z_0, z_1, \ldots, z_m) \in [0,b]\times E^{m+1}.$

(A_2) The function $F_2 : [0,b]\times E^{n+1} \to E$ meets the conditions stated below:

(i) For all $0 \le \iota \le b,$ the function $F_2(\iota,.) : E^{n+1} \to E$ is continuous and $\forall (z_0, z_1, \ldots, z_n) \in E^{n+1}$, the function $F_2(., z_0, z_1, \ldots, z_n) : [0,b] \to E$ is strongly measurable;

(ii) There is a positive function $h_r(.) : [0,b] \to R^{+}\ \forall r > 0\ , r \in N,$ so as

$$\sup_{\|z_0\|,\ldots,\|z_n\| \le r} \left\| F_2(\iota, z_0, z_1, \ldots, z_n)\right\| \le h_r(\iota),$$

the function $s \to (\iota - s)^{1-\beta} h_r(s) \in L^1([0,\iota], R^{+})$ and $\exists$ there exists a $\vartheta_1 > 0$ so as

$$\liminf_{r\to\infty} \frac{1}{r}\int_0^{\iota} (\iota - s)^{1-\beta}\ h_r(s)\, ds = \vartheta_1 < \infty, \iota \in [0,b],$$

(A_3) The function $F_3 : [0,b]\times E\times E\times E \to E$ meets the conditions mentioned below:

(i) The function $F_3(\iota,.,.,.) : E\times E\times E \to E$, $\forall \iota \in [0,b],$ and $\forall z, y \in E,$ the function $F_3(., z, y) : [0,b] \to E$ is strongly measurable;

(ii) for all positive number $r \in N,$ there is a positive function $\chi_r(.) : [0,b] \to R^{+}$ so as

$$\sup_{\|z\|\le r}\left\| F_3\left(s, z(s), \int_0^{s} k_1(s,\xi,z(\xi))\, d\xi, \int_0^{b} k_2(s,\xi,z(\xi))\, d\xi\right)\right\| \le \chi_r(\iota),$$

the function $s \to (\iota - s)^{1-\beta} \chi_r(s) \in L^1([0,\iota], R^{+}),$ and there exists a $\vartheta_2 > 0$ so as

$$\liminf_{r\to\infty} \frac{1}{r}\int_0^{\iota} (\iota - s)^{1-\beta}\ \chi_r(s)\, ds = \vartheta_2 < \infty, \iota \in [0,b].$$

(A_4) $\psi_i, d_j \in C([0,b];[0,b]), i = 1,2,\ldots,m, j = 1,2,\ldots,n;\ k \in C(Y;E)$, here and after this $Y = C([0,b];E)$, and k meets that:

(i) We have positive constants C and C' so that $\|k(z)\| \le C\|z\| + C', \ \forall z \in Y$;
(ii) k be a completely continuous map.

At the end of this part, we evoke the Sadoviskii's fixed point theorem [27] that is imposed to demonstrate the existence of the mild solution of equation (1).

Theorem 1.2 (see [27]). Assume that a condensing operator on a Banach space X is ϕ, this implies that, ϕ is continuous and have bounded sets into bounded sets, and $\mu(\phi(D)) \le \mu(D)$, $\forall$ bounded set D of E with $\mu(D) > 0$. If $\phi(Y) \subset Y$ for convex, closed and bounded set Y of E, then ϕ has a fixed point in E (where $\mu(.)$ represent the Kuratowski's measures of non-compactness).

2. EXISTENCE OF MILD SOLUTION

Definition 2.3 A function $z(.):[0,b] \to E$ which is continuous and it is revealed to be a mild solution of the nonlocal Cauchy problem (1), if the function $(\iota - s)^{\beta-1} AT_\beta(\iota - s) F_1(s, z(s), z(\psi_1(s)), \ldots, z(\psi_m(s))), 0 \le s < b$ is integrable on $[0,b)$ and the integral equation is verified which is mentioned just below:

$$\begin{aligned} z(\iota) &= S_\beta(\iota)\left[z_0 + F_1(0, z(0), z(\psi_1(0)), \ldots, z(\psi_m(0))) - k(z)\right] - F_1(\iota, z(\iota), z(\psi_1(\iota)), \ldots, z(\psi_m(\iota))) \\ &+ \int_0^\iota (\iota - s)^{\beta-1} AT_\beta(\iota - s) F_1(s, z(s), z(\psi_1(s)), \ldots, z(\psi_m(s))) ds \\ &+ \int_0^\iota (\iota - s)^{\beta-1} T_\beta(\iota - s) F_2(s, z(s), z(d_1(s)), \ldots, z(d_n(s))) ds \\ &+ \int_0^\iota (\iota - s)^{\beta-1} T_\beta(\iota - s)\left[F_3\left(s, z(s), \int_0^s k_1(s, \xi, z(\xi)) d\xi, \int_0^b k_2(s, \xi, z(\xi)) d\xi\right)\right] ds, \ 0 \le \iota \le b \end{aligned} \quad \ldots(4)$$

where $S_\beta(\iota) z = \int_0^\infty \zeta_\beta(\theta) S(\iota^\beta \theta) z d\theta, \ \ T_\beta(\iota) z = \beta \int_0^\infty \theta \zeta_\beta(\theta) S(\iota^\beta \theta) z d\theta$

with ζ_β be a probability density function expressed on $(0,\infty)$, that is $\zeta_\beta(\theta) \ge 0, \theta \in (0,\infty)$ and $\int_0^\infty \zeta_\beta(\theta) d\theta = 1$.

Remark. $\int_0^\infty \theta \zeta_\beta(\theta) d\theta = \frac{1}{\Gamma(1+\beta)}$.

Lemma 2.1 (see [28]) The operators $S_\beta(\iota)$ and $T_\beta(\iota)$ have the properties registered below:

(i) for any fixed point $z \in E, \|S_\beta(\iota) z\| \le M_1 \|z\|, \|T_\beta(\iota) z\| \le \frac{\beta M_1}{\Gamma(\beta+1)} \|z\|$;

(ii) $\{S_\beta(\iota), \iota \ge 0\}$ and $\{T_\beta(\iota), \iota \ge 0\}$ are two strongly continuous;

(iii) for all $\iota > 0, S_\beta(\iota)$ and $T_\beta(\iota)$ are also compact operator;

(iv) for any $z \in E, \alpha \in (0,1)$ and $\gamma \in (0,1)$, we have $AT_\beta(\iota) z = A^{1-\alpha} T_\beta(\iota) A^\alpha x$ and

$$\|A^\gamma T_\beta(\iota)\| \le \frac{\beta C_\gamma \Gamma(2-\gamma)}{\iota^{\beta\gamma} \Gamma(1+\beta(1-\gamma))}, 0 < \iota \le b.$$

Theorem 2.3 The system (1) has a mild solution if the assumptions $(A_1)-(A_4)$ are fulfilled and $z_0 \in E$, such that

$$M_0 = M_2\left[(M_1+1)M' + \frac{C_{1-\alpha} \Gamma(1+\alpha) b^{\alpha\beta}}{\alpha \Gamma(1+\alpha\beta)}\right] < 1 \quad \ldots(5)$$

and

$$M_1\left[M' M_3 + C + \frac{\beta \vartheta_1}{\Gamma(\beta+1)} + \frac{\beta \vartheta_2}{\Gamma(\beta+1)}\right] + M' M_3 + \frac{C_{1-\alpha} \Gamma(1+\alpha) b^{\alpha\beta} M_3}{\alpha \Gamma(1+\alpha\beta)} < 1, \quad \ldots(6)$$

where $M' = \|A^{-\alpha}\|$.

Proof: For simplicity, we rephrase that

$$\left(\iota, z(\iota), z(\psi_1(\iota)),\ldots, z(\psi_m(\iota))\right) = (\iota, w(\iota))$$

and

$$\left(\iota, z(\iota), z(d_1(\iota)),\ldots, z(d_n(\iota))\right) = (\iota, \mathring{\omega}(\iota))$$

Define the operator φ on Y by

$$(\varphi x)(\iota) = S_\beta(\iota)\left[z_0 + F_1(0, w(0)) - k(z)\right] - F_1(\iota, w(\iota)) + \int_0^\iota (\iota - s)^{\beta-1} AT_\beta(\iota - s) F_1(s, w(s))\,ds$$
$$+\int_0^\iota (\iota - s)^{\beta-1} T_\beta(\iota - s) F_2(s, \mathring{\omega}(s))\,ds$$
$$+\int_0^\iota (\iota - s)^{\beta-1} T_\beta(\iota - s)\left[F_3\left(s, z(s), \int_0^s k_1(s,\xi,z(\xi))\,d\xi, \int_0^b k_2(s,\xi,z(\xi))\,d\xi\right)\right]ds,\ 0 \le \iota \le b.$$

For all positive number $r,$ suppose

$$D_r = \left\{z \in Y : \|z(\iota)\| \le r,\ 0 \le \iota \le b\right\}.$$

Then $\forall r,\ D_r$ is explicitly a bounded closed convex set in $Y.$
By using Lemma 2.1, the equation (3) yields

$$\left\|\int_0^\iota (\iota - s)^{\beta-1} AT_\beta(\iota - s) F_1(s, w(s))\,ds\right\| \le \int_0^\iota \left\|(\iota - s)^{\beta-1} A^{1-\beta} T_\beta(\iota - s) A^\alpha F_1(s, w(s))\right\| ds$$
$$\le \frac{\beta C_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+\alpha\beta)} \int_0^\iota \frac{(\iota - s)^{\beta-1}}{(\iota - s)^{\beta-\alpha\beta}} \left\|A^\alpha F_1(s, w(s))\right\|$$
$$\le \frac{\beta C_{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1+\alpha\beta)} \int_0^\iota (\iota - s)^{\alpha\beta-1} \left\|A^\alpha F_1(s, w(s))\right\| ds$$
$$\le \frac{C_{1-\alpha}\Gamma(1+\alpha) b^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)} M_3\left(\left\{\|z_i\| : i = 0,\ldots,m\right\} + 1\right)$$
$$\le \frac{C_{1-\alpha}\Gamma(1+\alpha) a^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)} M_3(r+1) \qquad \text{...(7)}$$

then with the use of Bochner's theorem [29], it notes that $(\iota - s)^{\beta-1} AT_\beta(\iota - s) F_1(s, w(s))$ is integrable on $[0,b]$, therefore φ is well defined on D_r. In the same manner, from (A_2)(ii), we find

$$\left\|\int_0^\iota (\iota - s)^{\beta-1} T_\beta(\iota - s) F_2(s, \mathring{w}(s))\,ds\right\| \le \int_0^\iota \left\|(\iota - s)^{\beta-1} T_\beta(\iota - s) F_2(s, \mathring{w}(s))\right\| ds$$
$$\le \frac{\beta M_1}{\Gamma(\beta+1)} \int_0^\iota (\iota - s)^{\beta-1} \left\|F_2(s, \mathring{w}(s))\right\| ds$$
$$\le \frac{\beta M_1}{\Gamma(\beta+1)} \int_0^\iota (\iota - s)^{\beta-1} h_r(s)\,ds. \qquad \text{...(8)}$$

Further from (A_3) (ii), we get

$$\left\|\int_0^\iota (\iota - s)^{\beta-1} T_\beta(\iota - s)\left[F_3\left(s, z(s), \int_0^s k_1(s,\xi,z(\xi))\,d\xi, \int_0^b k_2(s,\xi,z(\xi))\,d\xi\right)\right]ds\right\|$$

$$\leq \int_0^{\iota} \left\| (\iota-s)^{\beta-1} T_\beta(\iota-s)\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))\,d\xi,\int_0^b k_2(s,\xi,z(\xi))\,d\xi\right)\right]\right\| ds$$

$$\leq \frac{\beta M_1}{\Gamma(\beta+1)}\int_0^{\iota}(\iota-s)^{\beta-1}\left\|F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))\,d\xi,\int_0^b k_2(s,\xi,z(\xi))\,d\xi\right)\right\| ds$$

$$\leq \frac{\beta M_1}{\Gamma(\beta+1)}\int_0^{\iota}(\iota-s)^{\beta-1}\chi_r(s)\,ds \qquad ...(9)$$

We assert that there is a positive number r to such an extent that $\varphi D_r \subseteq D_r$. If it is false, then $\forall$ positive number r, there exists a function $z_r(.)\in D_r$, but $\varphi z_r \notin D_r$, i.e. $\|\varphi z_r(\iota)\| > r$ for some $0\leq \iota(r)\leq b$, where $\iota(r)$ denotes ι isindependent of r. However, at the same time, we get

$$r \leq \|(\varphi z_r)(\iota)\|$$

$$\leq M_1\left[\|z_0\| + M'M_3(r+1) + (Cr+C')\right] + M'M_3(r+1) + \frac{C_{1-\alpha}\Gamma(1+\alpha)b^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)}M_3(r+1)$$

$$+\frac{\beta M_1}{\Gamma(\beta+1)}\int_0^{\iota}(\iota-s)^{\beta-1}h_r(s)\,ds + \frac{\beta M_1}{\Gamma(\beta+1)}\int_0^{\iota}(\iota-s)^{\beta-1}\chi_r(s)\,ds.$$

$$\leq M_1\left[\|z_0\| + M'M_3(r+1) + (Cr+C')\right] + M'M_3(r+1) + \frac{C_{1-\alpha}\Gamma(1+\alpha)a^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)}M_3(r+1)$$

$$+\frac{\beta M_1}{\Gamma(\beta+1)}\int_0^{b}(b-s)^{\beta-1}h_r(s)\,ds + \frac{\beta M_1}{\Gamma(\beta+1)}\int_0^{b}(b-s)^{\beta-1}\chi_r(s)\,ds. \qquad (10)$$

By dividing both sides of equation (10) by r and taking the lower limit as $r\to+\infty$, we find

$$1\leq M_1M'M_3 + M_1C + M'M_3 + \frac{C_{1-\alpha}\Gamma(1+\alpha)b^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)}M_3 + \frac{\beta M_1}{\Gamma(\beta+1)}\vartheta_1 + \frac{\alpha M}{\Gamma(\alpha+1)}\vartheta_2$$

Or $$M_1\left[M'M_3 + C + \frac{\beta}{\Gamma(\beta+1)}\vartheta_1 + \frac{\beta}{\Gamma(\beta+1)}\vartheta_2\right] + M'M_3 + \frac{C_{1-\alpha}\Gamma(1+\alpha)b^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)}M_3 \geq 1.$$

This contradicts (6). Hence, for positive r, $\varphi D_r \subseteq D_r$.

Further, we want to demonstrate that the operator φ has a fixed point on D_r, which means that (1) has a mild solution. At this point, we decompose φ like $\varphi=\varphi_1+\varphi_2$, in which the operators φ_1,φ_2 are defined on D_r, respectively, by

$$(\varphi_1 z)(\iota) = S_\beta(\iota)F_1(0,w(0)) - F_1(\iota,w(\iota)) + \int_0^{\iota}(\iota-s)^{\beta-1}AT_\beta(\iota-s)F_1(s,w(s))\,ds$$

and

$$(\varphi_2 z)(\iota) = S_\beta(\iota)\left[z_0 - k(z)\right] + \int_0^{\iota}(\iota-s)^{\beta-1}T_\beta(\iota-s)F_2(s,\mathring{w}(s))\,ds$$

$$+\int_0^{\iota}(\iota-s)^{\beta-1}T_\beta(\iota-s)\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))\,d\xi,\int_0^b k_2(s,\xi,z(\xi))\,d\xi\right)\right]ds,$$

for $0\leq\iota\leq b$, and we will demonstrate that φ_1 is a contraction as well as φ_2 is a compact operator.

For proving φ_1 is a contraction, we take $z_1,z_2\in D_r$, then $\forall\iota\in[0,b]$ and from assumption (A_1) and (5), we find

$$\|(\varphi_1 z_1)(\iota) - (\varphi_1 z_2)(\iota)\| \leq \left\|S_\beta(\iota)\left[F_1(0,w_1(0)) - F_1(0,w_2(0))\right]\right\|$$

$$+\left\|F_1(\iota,w_1(\iota))-F_1(\iota,w_2(\iota))\right\|+\left\|\int_0^\iota(\iota-s)^{\beta-1}AT_\beta(\iota-s)\left[F_1(s,w_1(s))-F_2(s,w_2(s))\right]ds\right\|$$

$$\le(M_1+1)M'M_2\sup_{0\le s\le b}\left\|z_1(s)-z_2(s)\right\|+\frac{C_{1-\alpha}\Gamma(1+\alpha)M_2b^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)}\sup_{0\le s\le b}\left\|z_1(s)-z_2(s)\right\|$$

$$\le M_2\left[(M_1+1)M'+\frac{C_{1-\alpha}\Gamma(1+\alpha)b^{\alpha\beta}}{\alpha\Gamma(1+\alpha\beta)}\right]\sup_{0\le s\le b}\left\|z_1(s)-z_2(s)\right\|$$

$$=M_0\sup_{0\le s\le b}\left\|z_1(s)-z_2(s)\right\|.$$

Hence $\left\|\varphi z_1-\varphi z_2\right\|\le M_0\sup_{0\le s\le b}\left\|z_1(s)-z_2(s)\right\|$.

So from assumption $0<M_0<1$, we observe that φ_1 is a contraction.

For proving φ_2 is a compact, first of all we establish that φ_2 is continuous on D_r. Consider that $\{z_n\}\subseteq D_r$ with $z_n\to z$ in D_r, then by assumptions (A_2)(i) and (A_3)(i), we get

$$F_2\left(s,\mathring{w}_n(s)\right)\to F_2\left(s,\mathring{w}(s)\right)\text{ as } n\to\infty.$$

$$F_3\left(\iota,z_n(\iota),\int_0^\iota k_1(\iota,s,z_n(s))ds,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)$$

$$\to F_3\left(\iota,z(\iota),\int_0^\iota k_1(\iota,s,z(s))ds,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\text{ as } n\to\infty.$$

Since

$$\left\|F_2\left(s,\mathring{w}_n(s)\right)-F_2\left(s,\mathring{w}(s)\right)\right\|\le 2h_r(s),$$

$$\Bigg\|F_3\left(\iota,z_n(\iota),\int_0^\iota k_1(\iota,s,z_n(s))ds,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)$$

$$-F_3\left(\iota,z(\iota),\int_0^\iota k_1(\iota,s,z(s))ds,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\Bigg\|\le 2\chi_r(s).$$

With the help of the dominated convergence theorem, we obtain

$$\left\|\varphi_2z_n-\varphi_2z\right\|=\sup_{0\le\iota\le b}\left\|\begin{array}{c}S_\beta(\iota)\left[k(z)-k(z_n)\right]\\ +\int_0^\iota(\iota-s)^{\beta-1}T_\beta(\iota-s)\left[F_2\left(s,\mathring{w}_n(s)\right)-F_2\left(s,\mathring{w}(s)\right)\right]ds\\ +\int_0^\iota(\iota-s)^{\beta-1}T_\beta(\iota-s)\left[\begin{array}{l}F_3\left(s,z_n(s),\int_0^s k_1(s,\xi,z_n(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\\ -F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\end{array}\right]ds\end{array}\right\|\to 0,$$

as $n\to\infty$, that is φ_2 is continuous.

Furthermore, we establish that $\{\varphi_2z:z\in D_r\}$ is a family of equicontinuous functions. To understand this we fix $\iota_1>0$ and consider $\iota_2>\iota_1$ and $\upsilon>0$, be enough small. So,

$$\left\|(\varphi_2z)(\iota_2)-(\varphi_2z)(\iota_1)\right\|\le\left\|S_\beta(\iota_2)-S_\beta(\iota_1)\right\|\left\|z_0-k(z)\right\|$$

$$+\int_0^{\iota_1-\upsilon}\left\|(\iota_2-s)^{\beta-1}T_\beta(\iota_2-s)-(\iota_1-s)^{\beta-1}T_\beta(\iota_1-s)\right\|\left\|F_2\left(s,\mathring{w}(s)\right)\right\|ds$$

$$+\int_{\iota_1-\upsilon}^{\iota_1}\left\|(\iota_2-s)^{\beta-1}T_\beta(\iota_2-s)-(\iota_1-s)^{\beta-1}T_\beta(\iota_1-s)\right\|\left\|F_2\left(s,\mathring{w}(s)\right)\right\|ds$$

$$+\int_{\iota_1}^{\iota_2}\left\|(\iota_2-s)^{\beta-1}T_\beta(\iota_2-s)\right\|\left\|F_2\left(s,\mathring{w}(s)\right)\right\|ds$$

$$+\int_0^{\iota_1-\upsilon}\left\|(\iota_2-s)^{\beta-1}T_\beta(\iota_2-s)-(\iota_1-s)^{\beta-1}T_\beta(\iota_1-s)\right\|\left\|F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right\|ds$$

$$+\int_{\iota_1-\upsilon}^{\iota_1}\left\|(\iota_2-s)^{\beta-1}T_\beta(\iota_2-s)-(\iota_1-s)^{\beta-1}T_\beta(\iota_1-s)\right\|\left\|F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right\|ds$$

$$+\int_{\iota_1}^{\iota_2}\left\|(\iota_2-s)^{\beta-1}T_\beta(\iota_2-s)\right\|\left\|F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right\|ds.$$

We observe that $\|(\varphi_2 z)(\iota_2)-(\varphi_2 z)(\iota_1)\|$ tends to zero independently of $z\in D_r$ as $\iota_2\to\iota_1$, with υ enough small since the compactness of $S_\beta(\iota)$ for $\iota>0$ (see [16]) means the continuity of $S_\beta(\iota)$ for $\iota>0$ in ι in the uniform operator topology. In the same fashion, using the compactness of the set $k(D_r)$ we can establish that the function $\varphi_2 z, z\in D_r$ are equicontinuous at $\iota=0$. Thus, φ_2 maps D_r into a family of eqicontinuous functions.

It residues to establish that $U(\iota)=\{(\varphi_2 z)(\iota):z\in D_r\}$ is relatively compact in E. $U(0)$ is relatively compact in E. Assume that $0<\iota\le b$ be fixed and $0<\upsilon<\iota$, arbitrary $\varepsilon>0$, for $z\in D_r$, we express

$$\left(\varphi_2^{\upsilon,\varepsilon}z\right)(\iota)=\int_\varepsilon^\infty\lambda_\beta(\theta)S\left(\iota^\beta\theta\right)\left[z_0-k(z)\right]d\theta+\beta\int_0^{\iota-\upsilon}\int_\varepsilon^\infty\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)F_2\left(s,\mathring{w}(s)\right)d\theta\, ds$$

$$+\beta\int_0^{\iota-\upsilon}\int_\varepsilon^\infty\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)\times$$

$$\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right]d\theta\, ds$$

$$=S\left(\upsilon^\beta\varepsilon\right)\int_\varepsilon^\infty\lambda_\beta(\theta)S\left(\iota^\beta\theta-\upsilon^\beta\varepsilon\right)\left[z_0-k(z)\right]d\theta$$

$$+\beta S\left(\upsilon^\beta\varepsilon\right)\int_0^{\iota-\upsilon}\int_\varepsilon^\infty\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta-\upsilon^\beta\varepsilon\right)F_2\left(s,\mathring{w}(s)\right)d\theta\, ds$$

$$+\beta S\left(\upsilon^\beta\varepsilon\right)\int_0^{\iota-\upsilon}\int_\varepsilon^\infty\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta-\upsilon^\beta\varepsilon\right)\times$$

$$\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right]d\theta ds$$

Seeing that $S\left(\upsilon^\beta\varepsilon\right),\upsilon^\beta\varepsilon>0$ is a compact operator, then the set $U^{\upsilon,\varepsilon}(\iota)=\left\{\left(\varphi_2^{\upsilon,\varepsilon}z\right)(\iota):z\in D_r\right\}$ is relatively compact in E $\forall\upsilon, 0<\upsilon<$ and $\forall\varepsilon>0$. Also, $\forall z\in D_r$, we obtain

$$\left\|(\varphi_2 z)(\iota)-(\varphi_2^{\upsilon,\varepsilon} z)(\iota)\right\| \le \left\|\int_0^{\varepsilon}\lambda_\beta(\theta)S(\iota^\beta\theta)[z_0-k(z)]d\theta\right\|$$

$$+\beta\left\|\int_0^{\iota}\int_0^{\varepsilon}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)F_2\left(s,\mathring{w}(s)\right)d\theta\,ds\right\|$$

$$+\beta\left\|\begin{array}{l}\int_0^{\iota}\int_{\varepsilon}^{\infty}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)F_2\left(s,\mathring{w}(s)\right)d\theta\,ds\\ -\int_0^{\iota-\upsilon}\int_{\varepsilon}^{\infty}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)F_2\left(s,\mathring{w}(s)\right)d\theta\,ds\end{array}\right\|$$

$$+\beta\left\|\int_0^{\iota}\int_0^{\varepsilon}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right]d\theta\,ds\right\|$$

$$+\beta\left\|\begin{array}{l}\int_0^{\iota}\int_{\varepsilon}^{\infty}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right]d\theta\,ds\\ -\int_0^{\iota-\upsilon}\int_{\varepsilon}^{\infty}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right]d\theta\,ds\end{array}\right\|$$

$$\le \left\|\int_0^{\varepsilon}\lambda_\beta(\theta)S(\iota^\beta\theta)[z_0-k(z)]d\theta\right\|$$

$$+\beta\left\|\int_0^{\iota}\int_0^{\varepsilon}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)F_2\left(s,\mathring{w}(s)\right)d\theta\,ds\right\|$$

$$+\beta\left\|\int_{\iota-\upsilon}^{\iota}\int_{\varepsilon}^{\infty}\theta(\iota-s)^{\varepsilon-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)F_2\left(s,\mathring{w}(s)\right)d\theta\,ds\right\|$$

$$+\beta\left\|\int_0^{\iota}\int_0^{\varepsilon}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right]d\theta\,ds\right\|$$

$$+\beta\left\|\int_{\iota-\upsilon}^{\iota}\int_{\varepsilon}^{\infty}\theta(\iota-s)^{\beta-1}\lambda_\beta(\theta)S\left((\iota-s)^\beta\theta\right)\left[F_3\left(s,z(s),\int_0^s k_1(s,\xi,z(\xi))d\xi,\int_0^b k_2(s,\xi,z(\xi))d\xi\right)\right]d\theta\,ds\right\|$$

$$\le M_1\left[\|z_0\|+C\|z\|+C'\right]\int_0^{\varepsilon}\lambda_\beta(\theta)d\theta$$

$$+\beta M_1\left(\int_0^{\iota}(\iota-s)^{\beta-1}h_r(s)ds\right)\int_0^{\varepsilon}\theta\lambda_\beta(\theta)d\theta+\beta M_1\left(\int_{\iota-\upsilon}^{\iota}(\iota-s)^{\beta-1}h_r(s)ds\right)\int_0^{\infty}\theta\lambda_\beta(\theta)d\theta$$

$$+\beta M_1\left(\int_0^{\iota}(\iota-s)^{\beta-1}\chi_r(s)ds\right)\int_0^{\varepsilon}\theta\lambda_\beta(\theta)d\theta+\beta M_1\left(\int_{\iota-\upsilon}^{\iota}(\iota-s)^{\beta-1}\chi_r(s)ds\right)\int_0^{\infty}\theta\lambda_\beta(\theta)d\theta$$

$$\le M_1\left[\|z_0\|+C\|z\|+C'\right]\int_0^{\varepsilon}\lambda_\beta(\theta)d\theta$$

$$+\beta M_1\left(\int_0^{\iota}(\iota-s)^{\beta-1}h_r(s)ds\right)\int_0^{\varepsilon}\theta\lambda_\beta(\theta)d\theta+\frac{\beta M_1}{\Gamma(\beta+1)}\left(\int_{\iota-\upsilon}^{\iota}(\iota-s)^{\beta-1}h_r(s)ds\right)$$

$$+\beta M_1\left(\int_0^{\iota}(\iota-s)^{\beta-1}\chi_r(s)\,ds\right)\int_0^{\varepsilon}\theta\lambda_\beta(\theta)\,d\theta+\frac{\beta M_1}{\Gamma(\beta+1)}\left(\int_{\iota-\upsilon}^{\iota}(\iota-s)^{\beta-1}\chi_r(s)\,ds\right).$$

So, the relatively compact sets are arbitrarily near to the set $U(\iota),\iota>0$. Thus, $U(\iota),\iota>0$ is also relatively compact in Banach space E.

Hence, in view of Arzela-Ascoli theorem φ_2 is a compact operator. All these evidence validate us to summarize that $\varphi=\varphi_1+\varphi_2$, is a condensing map D_r, and by the Sadovskii fixed point theorem, we have a fixed point $z(.)$ for φ on D_r. Hence, the problem (1) has a mild solution, and thus the proof is finished.

3. CONCLUSION

In the paper, we establish the existence of a mild solution of the functional integrodifferential equations of neutral type of fractional order. For this purpose, we have used the tools of the fractional power of operators and the Sadovskii's fixed point theorem for getting the main outcomes.

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