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Some Properties of Pseudo-Differential Operators Involving Fractional Fourier Transform

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Abstract

Some properties of pseudo-differential operators on Schwartz space $S(\mathrm{R}^n)$ are studied by using the fractional Fourier transform.

1. Introduction

The concept of pseudo-differential operators was originated by Kohn-Nirenberg [1], Hörmander [2, 3] and others. Pseudo-differential operators on $S(\mathbb{R})$ have been discussed in association with fractional Fourier transform by Pathak et al. [4], Prasad and Kumar [5]. In this connection, pseudo-differential operators of infinite order involving fractional Fourier transform on $W_M(\mathbb{R}^n)$ and $W_M^{\Omega}(\mathbb{C}^n)$ have been studied by Upadhyay et al. [6], Upadhyay and Dubey [7] respectively. The main aim of this paper is to discuss the some properties of pseudo-differential operators involving fractional Fourier transform on Schwartz space $S(\mathbb{R}^n)$, where $\mathbb{R}^n$ is usual Euclidean space. If $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$are the elements of $\mathbb{R}^n$.

Then the inner product of *x* and *y* is defined by

$$\langle x,y\rangle = x.y = \sum_{j=1}^{n} x_j \,.\, y_j \tag{1}$$

and the norm of *x* is defined by

$$|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} \tag{2}$$

If $\beta = (\beta_1,\ldots,\beta_n)$ is an n-tuple of non-negative integers, then β is called a multi-indices. We write, $|\beta| = \beta_1 + \cdots + \beta_n$ and for $x \in \mathbb{R}^n, x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$. The *n*-dimensional fractional Fourier transform with parameter α of $\phi(x)$ on $x \in \mathbb{R}^n$ is denoted by $(F_\alpha\phi)(\xi) = \hat{\phi}_\alpha(\xi)$ [8, 6] and defined as

$$\hat{\phi}_\alpha(\xi) = (F_\alpha\phi)(\xi) = \int_{\mathbb{R}^n} K_\alpha(x,\xi)\phi(x)dx, \xi \in \mathbb{R}^n \tag{3}$$

where

$$K_\alpha(x,\xi) = \begin{cases} C_\alpha e^{\frac{i(|x|^2+|\xi|^2)\cot\alpha}{2} - i\langle x,\xi\rangle \csc\alpha} & if\ \alpha \neq n\pi \\ \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-i\langle x,\xi\rangle} & if\ \alpha = \frac{\pi}{2}, \end{cases} \quad \forall \mathrm{n} \in \mathbb{Z},$$

and

$$C_\alpha = (2\pi i \sin\alpha)^{-\frac{n}{2}} e^{\frac{in\alpha}{2}}$$

The corresponding inversion formula is given by

$$\phi(x) = \int_{\mathbb{R}^n} \overline{K_\alpha(x,\xi)}\, \hat{\phi}_\alpha(\xi)d\xi, x \in \mathbb{R}^n \tag{4}$$

where the kernel

$$\overline{K_\alpha(x,\xi)} = C'_\alpha e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2}+i\langle x,\xi\rangle \csc\alpha},$$

and

$$C'_\alpha = \frac{(2\pi i \sin\alpha)^{\frac{n}{2}}}{(2\pi \sin\alpha)^n} e^{\frac{-in\alpha}{2}}. \tag{5}$$

Definition 1.1 The Schwartz space $S(\mathbb{R}^n)$ is the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\gamma_{\mu,v}(\phi) = \sup_{x\in\mathbb{R}^n} |x^\mu D^v \phi(x)| < \infty, \tag{6}$$

for all multi-indices μ and v.

Definition 1.2 Let $m \in \mathbb{R}$. Then we define the symbol class S^m to be the space of all $\theta(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any two multi-indices μ and v, there is a positive constant $C_{\mu,v}$ depending upon μ and v such that

$$\left|\left(D_x^\mu D_\xi^v\right)(x,\xi)\right| \le C_{\mu,v}(1+|\xi|)^{m-|v|}, \qquad \forall\, x,\xi \in \mathbb{R}^n. \tag{7}$$

Definition 1.3 Let $\theta(x,\xi)$ be a symbol belonging to S^m, then the pseudodifferential operator $A_{\theta,\alpha}$ associated with $\theta(x,\xi)$ is defined as

$$(A_{\theta,\alpha}\phi)(x) = \int_{\mathbb{R}^n} \overline{K_\alpha(x,\xi)}\, \theta(x,\xi)\hat{\phi}_\alpha(\xi)d\xi, \phi \in S(\mathbb{R}^n) \tag{8}$$

where $\hat{\phi}_\alpha(\xi)$ is the fractional Fourier transform of $\phi(x)$, defined by (3), and

$$\overline{K_\alpha(x,\xi)} = C'_\alpha e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2}+i\langle x,\xi\rangle \csc\alpha}, \tag{9}$$

where C'_α is defined by (5).

2. Pseudo-Differential Operators Involving Fractional Fourier Transform

In this section, some properties of pseudo-differential operator associated with fractional Fourier transform defined by (8) on $S(\mathbb{R}^n)$ are discussed.

Theorem 2.1 Let $\theta(x,\xi) \in S^m$, where $m \in \mathbb{R}$. Then $A_{\theta,\alpha}$ maps $S(\mathbb{R}^n)$ into itself.

Proof. Let $\phi \in S(\mathbb{R}^n)$. Then for any two multi-indices μ and v, we need only prove that

$$\sup_{x\in\mathbb{R}^n} \left|x^\mu D_x^v (A_{\theta,\alpha}\phi)(x)\right| < \infty. \tag{10}$$

Proof of the above theorem follows from the facts of Pathak et. al. [4, Theorem 4.1] and Wong [9, pp. 31-32].

Theorem 2.2 Let $\theta(\xi) \in C^k(\mathbb{R}^n), k > \frac{n}{2}$, be such that there is a positive constant $C_{\beta,n}$ depending on β and n only, such that

$$\left|\left(D_\xi^\beta \theta\right)(\xi)\right| \le C_{\beta,n}(1+|\xi|)^{-\beta} \tag{11}$$

for multi-indices β with $|\beta| \le k$. Then for $1 \le p < \infty$, there exists a positive constant $B'_{\alpha,\beta,n}$ depending on α,β and n only, we have

$$\left\|(A_{\theta,\alpha}\phi)(x)\right\|_p \le B'_{\alpha,\beta,n} \|\phi(x)\|_p \;\forall \phi(x) \in S(\mathbb{R}^n), \tag{12}$$

where

$$(A_{\theta,\alpha}\phi)(x) = C'_\alpha \int_{\mathbb{R}^n} e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2}+i\langle x,\xi\rangle \csc\alpha}\, \theta(\xi)\hat{\phi}_\alpha(\xi)d\xi. \tag{13}$$

Proof. From (13, we can write

$$(A_{\theta,\alpha}\phi)(x) = F_\alpha^{-1}\left[\theta(\xi)\hat{\phi}_\alpha(\xi)\right](x). \tag{14}$$

Now we assume that

$$F_\alpha^{-1}\left[\theta(\xi)\hat{\phi}_\alpha(\xi)\right](x) = (f * g)(x).$$

Then,

$$\theta(\xi)\hat{\phi}_\alpha(\xi) = F_\alpha[(f * g)(x)](\xi),$$
$$= C_\alpha \int_{\mathbb{R}^n} e^{\frac{i(|x|^2+|\xi|^2)\cot\alpha}{2} - i\langle x,\xi\rangle \csc\alpha}(f * g)(x)dx.$$

From the arguments of [7, pp. 121, 122], we obtain

$$\theta(\xi)\hat{\phi}_\alpha(\xi) = \frac{1}{C_\alpha} \times e^{\frac{-i|\xi|^2 \cot x}{2}} \hat{f}_\alpha(\xi)\hat{g}_\alpha(\xi). \tag{15}$$

From (15), we get

$$\theta(\xi) = \frac{1}{C_\alpha} \times e^{\frac{-i|\xi|^2 \cot\alpha}{2}} \hat{f}_\alpha(\xi), \;\; \hat{\phi}_\alpha(\xi) = \hat{g}_\alpha(\xi). \tag{16}$$

Therefore,

$$f(x) = C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right](x), \;\; g(x) = \phi(x). \tag{17}$$

Thus, the (14) gives

$$\left(A_{\theta,\alpha}\phi\right)(x) = \left(C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right] * \phi\right)(x) \tag{18}$$

Using convolution property $\|f * \phi\|_p \le \|f\|_1\|\phi\|_p$ for $f \in L^1(\mathbb{R}^n)$ and $\phi \in L_p(\mathbb{R}^n)$, we have

$$\left\|\left(A_{\theta,\alpha}\phi\right)(x)\right\|_p = \left\|\left(C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right] * \phi\right)(x)\right\|_p$$
$$\le \left\|C_\alpha F_\alpha^{-1}\left[e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right](x)\right\|_1 \|\phi(x)\|_p. \tag{19}$$

Now, we shall prove that

$$C_\alpha F_\alpha^{-1}\left(e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right)(x) \in L^1(\mathbb{R}^n). \tag{20}$$

Thus,

$$F_\alpha^{-1}\left(e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right)(x) = \frac{C'_\alpha}{C_\alpha}\int_{\mathbb{R}^n} e^{\frac{-i(|x|^2+|\xi|^2)\cot\alpha}{2} + i\langle x,\xi\rangle\csc\alpha} e^{\frac{i|\xi|^2 \cot x}{2}}\theta(\xi)\, d\xi$$
$$= \frac{C'_\alpha}{C_\alpha}\int_{\mathbb{R}^n} e^{\frac{-i|x|^2 \cot\alpha}{2} + i\langle x,\xi\rangle\csc\alpha}\theta(\xi)\, d\xi.$$

Then

$$x^\beta F_\alpha^{-1}\left(e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right)(x) = \frac{C'_\alpha}{C_\alpha}(i\csc\alpha)^{-|\beta|} e^{\frac{-i|x|^2 \cot\alpha}{2}} \int_{\mathbb{R}^n} D_\xi^\beta\left(e^{i\langle x,\xi\rangle\csc\alpha}\right)\theta(\xi)\, d\xi.$$

By using the parts of integration, we get

$$x^\beta F_\alpha^{-1}\left(e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right)(x) = \frac{C'_\alpha}{C_\alpha}(-1)^{|\beta|}(i\csc\alpha)^{-|\beta|} e^{\frac{-i|x|^2 \cot\alpha}{2}} \int_{\mathbb{R}^n} \left(e^{i\langle x,\xi\rangle\csc\alpha}\right)(D_\xi^\beta\theta)(\xi)\, d\xi.$$

Therefore,

$$\left|x^\beta F_\alpha^{-1}\left(e^{\frac{i|\xi|^2 \cot\alpha}{2}}\theta(\xi)\right)(x)\right| \le \left|\frac{C'_\alpha}{C_\alpha}(\csc\alpha)^{-|\beta|}\right| \int_{\mathbb{R}^n} \left|\left(e^{i\langle x,\xi\rangle\csc\alpha}\right)\right| \left|(D_\xi^\beta\theta)(\xi)\right| d\xi.$$

Using (11), we have

$$\left|x^{\beta}F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^{2}\cot\alpha}{2}}\theta(\xi)\right)(x)\right| \leq \left|\frac{C'_{\alpha}}{C_{\alpha}}(\csc\alpha)^{-|\beta|}\right| C_{\beta,n}\int_{\mathbb{R}^n}(1+|\xi|)^{-|\beta|}\,d\xi.$$

Thus the last integral is convergent for sufficiently large β, then we get

$$\left|x^{\beta}F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^{2}\cot\alpha}{2}}\theta(\xi)\right)(x)\right| \leq \left|\frac{C'_{\alpha}}{C_{\alpha}}(\csc\alpha)^{-|\beta|}\right| C_{\beta,n}C_{\beta}.$$

Hence, we have

$$\left|F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^{2}\cot\alpha}{2}}\theta(\xi)\right)(x)\right| \leq C_{\beta,n,\csc\alpha}(1+|x|)^{-|\beta|}.$$

Then,

$$\begin{aligned}\left\|F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^{2}\cot\alpha}{2}}\theta(\xi)\right)(x)\right\|_{1} &\leq C_{\beta,n,\csc\alpha}\left\|(1+|x|)^{-|\beta|}\right\|_{1}\\ &< \infty.\end{aligned}$$

Therefore,

$$F_{\alpha}^{-1}\left(e^{\frac{i|\xi|^{2}\cot\alpha}{2}}\theta(\xi)\right)(x) \in L^{1}(\mathbb{R}^n).$$

From (19), we get

$$\begin{aligned}\left\|(A_{\theta,\alpha}\phi)(x)\right\|_{p} &\leq C_{\beta,n,\csc\alpha}\left\|(1+|x|)^{-|\beta|}\right\|_{1}\|\phi(x)\|_{p}\\ &\leq B'_{\alpha,\beta,n}\|\phi(x)\|_{p}, \text{ for } (x)\in S(\mathbb{R}^n).\end{aligned}$$

A similar theorem has been studied by Upadhyay et al. [10] on $W_M(\mathbb{R}^n)$ space by using Fourier transform.

Example 2.1 Consider a generalized differential operator, which is defined by $\triangle_x^s = \triangle_{x_1}^{s_1} \cdots \triangle_{x_n}^{s_n}$, where $s \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$ and for each j = 1, 2, · · · n, we have

$$\triangle_{x_j}^{s_j} = \left(-i\frac{\partial}{\partial x_j} + x_j\cot\alpha\right)^{s_j}.$$

Let $\phi \in S\,(\mathbb{R}^n)$, then using (4) we have

$$\begin{aligned}\triangle_x^s\,\phi(x) &= \triangle_x^s \int_{\mathbb{R}^n}\overline{K_{\alpha}(x,\xi)}\,\hat{\phi}_{\alpha}(\xi)d\xi, \qquad \forall\xi\in\mathbb{R}^n\\ &= \left(-i\frac{\partial}{\partial x} + x\cot\alpha\right)^{s} C'_{\alpha}\int_{\mathbb{R}^n} e^{\frac{-i(|x|^2+|\xi|^2)\cot x}{2} + i\langle x,\xi\rangle\csc\alpha}\hat{\phi}_{\alpha}(\xi)d\xi\\ &= \int_{\mathbb{R}^n}\overline{K_{\alpha}(x,\xi)}\,(\xi\csc\alpha)^{s}\hat{\phi}_{\alpha}(\xi)d\xi.\end{aligned}$$

Since $(\xi\ \csc\alpha)^s \in S^m$ for $m = |s| = s_1 + \cdots s_n$ where $m \in \mathbb{Z}_+$. Hence generalised differential operator $\triangle_x^s$ is a pseudo-differential operator with symbol $(\xi\ \csc\alpha)^s$ in the sense of fractional Fourier transform.

References

[1] J.J. Kohn and N. Nirenberg, "On the algebra of pseudo-differential operators," Comm. Pure Appl. Math 18, 269-305, 1965.

[2] L. Hörmander, "Linear Partial Differential Operators," Springer-Verlag Berlin Heidelberg, New York, 1976.

[3] L. Hörmander, "Linear Partial Differential Operators," Actes, Congres intern. math. Tome 1, p. 121 à 133, 1970.

[4] R.S. Pathak, A. Prasad and M. Kumar, "Fractional Fourier transform of tempered distributions and generalized pseudo-differential operators," Journal of Pseudo-Differential Operators and Applications 3, 239-254, 2012.

[5] A. Prasad and M. Kumar, “Product of two generalized pseudo-differential operators involving fractional Fourier transform,” Journal of Pseudo-Differential Operators and Applications, 2, 355-365, 2011.

[6] S.K. Upadhyay, A. Kumar and J.K. Dubey, “Characterization of spaces of type W and pseudo-differential operators of infinite order involving fractional Fourier transform, Journal of Pseudo-Differential Operators and Applications, 5(2), pp. 215-230, 2014.

[7] S.K. Upadhyay and J.K. Dubey, “Pseudo-differential operators of infinite order on $W_M^{\Omega}(\mathbb{C}^n)$-space involving fractional Fourier transform,” Journal of Pseudo-Differential Operators and Applications, 6, pp. 113-133, 2015.

[8] H. De Bie, N. De Schepper, “Fractional Fourier transforms of hyper complex signals,” SIVip, 6, 381-388, 2012.

[9] M.W. Wong, “An introduction to pseudo-differential operators” 3rd edn. World Scientific, Singapore, 2014.

[10] S.K. Upadhyay, R.N. Yadav and L. Debnath, “Infinite pseudo-differential operators on $W_M(\mathbb{R}^n)$ space” Analysis, International mathematical journal of analysis and its applications, 32, pp. 163-178, 2012.

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