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Parameter Augmentation For Basic Hypergeometric Series By Cauchy Operator

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Abstract

In paper [1], certain new summation and transformation formulae for basic hypergeometric series by using the Cauchy augmentation operator are established. In the present paper, we shall deal with parameter augmentation by using Cauchy augmentation operator in some more known identities and establish some new results.

1. Introduction

Throughout this paper, we adopt the following notations and definitions [2]. Let $|q| < 1$ and the q-shifted factorial be defined by

$$(a)_\infty = (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n). \tag{1}$$

Clearly,

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}. \tag{2}$$

and the generalized basic hypergeometric series is defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, & a_2, & ..a_{r+1} \\ & & \\ b_1, & b_2, & ..\ b_r \end{matrix}; q, \ z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n..(a_{r+1})_n}{(q)_n(b_1)_n..(b_r)_n}z^n$$

where $|z| < 1$, $|q| < 1$.

We also recall that the q-binomial coefficient is defined by

$$\left(\frac{n}{k}\right) = \frac{(q)_n}{(q)_k(q)_{h-k}}$$

Chen et. al. [3] defined the following operator as

$$T(a,b;D_q) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(bD_q)^n \tag{3}$$

and termed the operator as Cauchy augmentation operator. They have used (3) to obtain an extensions of the Askey-Wilson and the Askey-Roy integrals and Sear's two term summation formula. Recently, Ali and Agnihotri [1] have also established some new identities of basic hypergeometric series using the Cauchy augmentation operator (3). It is easy to see that this operator possess the following basic property

$$T(a,b;D_q)\frac{1}{(ct;q)_\infty} = \frac{(abt;q)_\infty}{(bt,ct;q)_\infty}. \tag{4}$$

where

$$D_q f(a) = \frac{f(a)-f(aq)}{a}. \tag{5}$$

is the q-differential operator or q-derivative [4].

The operator (3) is a generalization of the following operator earlier introduced by Chen and Liu [5]

$$T(bD_q) = \sum_{n=0}^{\infty}\frac{(bD_q)^n}{(q;q)_n}. \tag{6}$$

Using (6), Chen and Liu [5, 6] have derived a number of known q-identities from their special cases. Zhang and Wang [7] have used the operator (6) for obtaining two new operator identities and established some q-series identities.

In the present work, we have used Cauchy augmentation operator (3) to produce some interesting summation and transformation formulae for basic hypergeometric series.

We shall also use the following transformation of ${}_3\varphi_2$ series [2]

$$ {}_3\phi_2\left(\begin{matrix} a & b & c \\ & d & e \end{matrix};q,\ \tfrac{de}{abc}\right) = \frac{(b,\, de/ab,\, de/bc)_\infty}{(d,\, e,\, de/abc)_\infty}\, {}_3\phi_2\left(\begin{matrix} d/b, & e/b, & de/abc \\ & de/ab, & de/bc \end{matrix};q,\ \tfrac{e}{a}\right) \tag{7} $$

the q-Pfaff-Saalschutz sum [2]

$$ {}_3\phi_2\left(\begin{matrix} a, & b, & q^{-n} \\ & c, & abc^{-1}q^{1-n} \end{matrix};q,\ q\right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \tag{8} $$

and the following sum [1]

$$ {}_3\phi_2\left(\begin{matrix} a, & b, & e \\ & aq, & de \end{matrix};q,\ q/b\right) = \frac{(q, aq/b, deq, e)_\infty}{(eq, aq, q/b, de)_\infty}. \tag{9} $$

2. Main Results

Theorem 2.1

We have

$$ {}_4\phi_3\left(\begin{matrix} a, & b, & c & g \\ & aq, & e, & fg \end{matrix};q,\ eq/bc\right) = \frac{(b,\, eq/b,\, aeq/bc,\, g,\, fgq/b)_\infty}{(e,\, eq/bc,\, fg,\, gq/b,\, aq)_\infty}\, {}_4\phi_3\left(\begin{matrix} aq/b, & e/b, & eq/bc, & gq/b \\ & eq/b, & aeq/bc, & fgq/b \end{matrix};q,\ b\right) \tag{10} $$

Proof : Taking $d = aq$ in (7), we obtain

$$ \sum_{n=0}^{\infty} \frac{(a)_\infty (b)_n (c)_n}{(q)_n (aq)_n (e)_n (aq^n)_\infty} (eq/bc)^n = \frac{(b,\, eq/b,\, aeq/bc)_\infty}{(aq,\, e,\, eq/bc)_\infty} \sum_{n=0}^{\infty} \frac{(e/b)_n (aq/b)_\infty (eq/bc)_n}{(q)_n (aeq/bc)_n (eq/b)_n (aq^{n+1}/b)_\infty} b^n. $$

Applying the operator $T(f, g; Dq)$ on both sides with respect to a and using (4), after some simplification we obtain (10).

Theorem 2.2

We have

$$ {}_4\phi_3\left(\begin{matrix} a, & b, & q^{-n}, & e \\ & aq, & bq^{-n}, & de \end{matrix};q,\ q\right) = \frac{(q,\, aq/b)_n (e,\, deq)_\infty}{(aq,\, q/b)_n (eq,\, de)_\infty} $$

Proof: Putting $c = aq$ in (8), we obtain

$$ \sum_{n=0}^{\infty} \frac{(a)_\infty (b)_n (q^{-n})_n}{(q)_n (aq)_n (b/q^n)_n (aq^n)_\infty} q^n = \frac{(q,\, aq/b)_n}{(q/b)_n} \frac{(aq^{n+1})_\infty}{(aq)_\infty}. \tag{11} $$

Applying the operator $T(d, e; Dq)$ on both sides with respect to a and using (4), we obtain (11).

Theorem 2.3.

We have

$$ {}_4\phi_3\left(\begin{matrix} a, & b, & e, & g \\ & aq, & de, & fg \end{matrix};q,\ q/b\right) = \frac{(q,\, aq/b,\, deq,\, e,\, fgq,\, g)_\infty}{(eq,\, q/b,\, de,\, aq,\, gq,\, fg)_\infty} \tag{2.3} $$

Proof: Equation (9) can be written as

$$(a)_\infty \sum_{n=0}^{\infty} \frac{(b)_n (e)_n}{(q)_n (aq)_n (de)_n (aq^n)_\infty} \left(\frac{q}{b}\right)^n = \frac{(q, aq/b, deq, e)_\infty}{(eq, q/b, de)_\infty} \frac{1}{(aq)_\infty}.$$

Applying the operator $T\,(f, g; Dq)$ and using (4), we have

$$(a)_\infty \sum_{n=0}^{\infty} \frac{(b)_n (e)_n (q/b)^n}{(q)_n (aq)_n (de)_n} \frac{(fgq^n)_\infty}{(aq^n)_\infty (gq^n)_\infty} = \frac{(q, aq/b, deq, e)_\infty}{(eq, q/b, de)_\infty} \frac{(fgq)_\infty}{(aq)_\infty (gq)_\infty}.$$

which on simplification yields (12).

3. Conclusion

The parameter augmentation is a well known classical and useful technique for the development of the transformations and summations for both ordinary and basic hypergeometric series as one can reach from fewer parameters to higher pa- rameters by applying this technique. In this paper, some new transformations and summation identities of basic hypergeometric series have been obtained by using Cauchy augmentation operator. More identities may be find by using Cauchy aug- mentation operator in some other known results.

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