2-Divisor Lucky Labeling of some Identity Graphs of Finite Group and some Zero-Divisor Graphs

K. Aruna Sakthi, R. Rajeswari, N. Meenakumari

1 A.P.C.Mahalaxmi College for Women, Thoothukudi, Affiliated to Manonmaniam Sundaranar University, Tirunelvel, Tamil Nadu, India.

2 PG and Research Department of Mathematics, A.P.C.Mahalaxmi College for Women, Thoothukudi, Tamil Nadu, India.

*Corresponding author Email: arunasakthi9397@gmail.com

Received: 15 March, 2022; Revised: 12 October, 2022; Accepted: 09 January, 2023; Published: 24 May, 2023

1. Introduction

The zero-divisor graph has two types. One is in the Beck definition (1988) in which all the elements in the rings will be considered for the vertex set of the graph. Other is in the Anderson and Livingston (1999) definition he slightly varied by considering only zero-divisor for the vertex set in the year 1999. Identity graphs, semigroups and some special subgraphs was studied by Kandasamy and Smarandache in this paper (6). Rosa (8) was the one who introduced graph labeling in the year 1967. Labeling (5) has many applications in the field of Engineering and technology etc. Lucky labeling was studied by Ahai et.al and Akbari et.al (1) (2). Applications of lucky labeling is in transportation network, to model protein structure etc. Inspiring d-lucky labeling and various type of lucky labeling 2-divisor lucky labeling has been introduce in this paper and studied for some identity graphs of finite groups and some zero-divisor graphs (9) (7).

2. Preliminaries

Definition 2.1: Zero-Divisor Graph:

Let R be a commutative ring with identity 1 and let Z(R) be its set of zero-divisors. We associate a graph Γ(R) to R with vertices, Z* = Z(R) – {0}, the set nonzero zero-divisor of R, and for distinct x, y ∈ Z(R)*, the vertices x and y are adjacent if and only if xy = 0. We denote their zero-divisor graph of R by Γ(R). If we take vertex set as Z(R). In Γ(R), the vertex 0 is adjacent to every other vertex. Γ(R) is an induced subgraph of Γ(R).

Definition 2.2: Identity Graph:

Let G be a group. The identity graph $\Gamma = (V, E)$ with vertices as the elements of group and two elements $x, y \in G$ are adjacent or can be joined by an edge if $xy = e$, where e is the identity element of G and identity element is adjacent to every other vertices in G.

3. 2-DIVISOR LUCKY LABELING FOR SOME IDENTITY GRAPHS

Definition: 2-Divisor Lucky Labeling:

A graph $G = (V, E)$ be a graph with n vertices and m edges. A graph G admits 2-divisor lucky labeling if $f : V(G) \rightarrow \{1, 2, ..., n\}$ be a labeling of vertices of graph G from $\{1, 2, 3, ..., n\}$. Define $s(v) = \left\lfloor \frac{|N(v)|}{2} \right\rfloor$, where $N(v)$ is the neighborhood of v such that $s(u) \neq s(v)$ for every pair of adjacent vertices u and v in G. The vertex 2-divisor lucky number is the least number from the set $\{1, 2, ..., n\}$ that has been used to label the graph G. It is denoted by $\eta_{vd}$. In this paper we have investigated for some types of identity graphs of finite group and some zero-divisor graphs.

Theorem: 3.1 2-divisor lucky number for the identity graph of for n be an odd number is two.

Proof: Let $G = \text{graph of } Z_n, n > 3$ Identity graph be an odd number.
2-Divisor Lucky Labeling of some Identity Graphs of Finite Group and some Zero-Divisor Graphs

\[ V(G) = \{0, 1, 2, \ldots, n - 1\} = \left\{ t_0, t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{2n-1} \right\}. \]

\[ E(G) = \left\{ t_0t_i, t_1t_{n-1}, t_2t_{n-2}, \ldots, t_{n-1}t_2, t_{n+1}t_2, \ldots, t_{2n-1}t_2 \right\}, \quad 1 \leq i \leq n - 1. \quad |V(G)| = n; \quad |E(G)| = \frac{3n-3}{2} \]

Define \( g: V(G) \to \{1, 2, \ldots, n\} \) such that \( g(t_0) = 1, g(t_1) = g(t_2) = \cdots = g\left(\frac{t_{n-1}}{2}\right) = 1 \) and \( g\left(\frac{t_{n+1}}{2}\right) = g\left(\frac{t_{n+3}}{2}\right) = \cdots = g(t_{n-1}) = 2 \).

\[
\begin{align*}
  s(t_0) &= \frac{g(t_1) + g(t_2) + \cdots + g(t_{n-1})}{2} \\
  s(t_1) &= \frac{g(t_0) + g(t_{n-1})}{2} \\
  s(t_2) &= \frac{g(t_0) + g(t_{n-2})}{2} \\
  s\left(\frac{t_{n-1}}{2}\right) &= \frac{g(t_0) + g\left(\frac{t_{n+1}}{2}\right)}{2} \\
  s(t_{n-1}) &= \frac{g(t_0) + g(t_1)}{2}
\end{align*}
\]

such that \( s(t_0) \neq s(t_1) \neq s(t_2) \neq \cdots \neq s\left(\frac{t_{n-1}}{2}\right) \neq s(t_{n-1}) \).

To label this Graph, only two labels have been used and also admits 2-divisor lucky labeling. Therefore, 2-divisor lucky number is two i.e \( \eta_{vd}(G) = 2 \).

**Theorem:** 3.2 The identity graph of \( (Z_n, \oplus_n) \) for \( n > 2 \) be an even number has 2-divisor lucky number to be two.

**Proof:** Let graph \( G = Identity\ graph\ of\ Z_n; n > 2 \) be an even number.

\[ V(G) = \{0, 1, 2, \ldots, n - 1\} = \left\{ t_0, t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{2n-1} \right\}. \]

\[ E(G) = \left\{ t_0t_i, t_1t_{n-1}, t_2t_{n-2}, \ldots, t_{n-1}t_2, t_{n+1}t_2, \ldots, t_{2n-1}t_2 \right\}, \quad 1 \leq i \leq n - 1. \quad |V(G)| = n; \quad |E(G)| = \frac{3n-3}{2} \]

Define \( g: V(G) \to \{1, 2, \ldots, n\} \) such that \( g(x_0) = 1 \), and \( g(x_i) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases} \)

\[ 1 \leq i \leq n - 1 \text{ and } i \neq \frac{n}{2} \quad g\left(\frac{x_n}{2}\right) = 2. \]
such that \( s(t_0) \neq s(t_1) \neq s(t_2) \neq \cdots \neq s\left(\frac{t_{n-1}}{2}\right) \neq s(t_{n-1}). \)

Graph \( G \) admits 2-divisor lucky labeling and to label this graph only two labels has been used. Therefore 2-divisor lucky number is two i.e \( \eta_{vd}(G)=2. \)

**Theorem: 3.3 Identity graph of Klein-4 group under composition has to be one.**

**Proof:** Let graph \( G = \) Identity graph of Klein-4 group

\[
V(G) = \{t_0, t_1, t_2, t_3\} = \{e, a, b, ab\}. \quad E(G) = \{t_0 t_i / 1 \leq i \leq 3\}.
\]

Define \( g: V(G) \rightarrow \{1, 2, 3, 4\} \) such that \( g(t_i) = 1 \) for all \( 1 \leq i \leq 4. \)

\[
s(t_0) = \left[\frac{g(t_1) + g(t_2) + g(t_3) + g(t_4)}{2}\right]
\]

\[
s(t_i) = \left[\frac{g(t_0)}{2}\right] \quad \text{for all } 1 \leq i \leq 3.
\]

such that \( s(t_0) \neq s(t_i) \) for all \( 1 \leq i \leq 3. \)

Only one label has been used to label this graph \( G. \) Therefore \( n_{vd}(G) \) is 1.

**Theorem: 3.4 The 2-divisor lucky number for the identity graph of quaternion group is two.**

**Proof:** Let graph \( G = \) identity graph of \( Q_8. \)

\[
V(G) = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}.
\]

\[
E(G) = \{t_0 t_i, t_2 t_3, t_4 t_5, t_6 t_7 : 1 \leq i \leq 7\}.
\]
Define \( g: V(G) \rightarrow \{1,2,3,...,8\} \) such that, \( g(t_0) = 1, g(t_i) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} ; 1 \leq i \leq 7. \)

\[
s(t_0) = \left[ \frac{g(t_1) + g(t_2) + g(t_3) + g(t_4) + g(t_5) + g(t_6) + g(t_7)}{7} \right]
\]

\[
s(t_1) = \left[ \frac{g(t_0)}{2} \right]
\]

\[
s(t_i) = \left[ \frac{g(t_0) + g(t_{i+1})}{2} \right] \quad \text{if } i = \text{even}
\]

\[
s(t_i) = \left[ \frac{g(t_0) + g(t_{i+1})}{2} \right] \quad \text{if } i = \text{odd except } i = 1
\]

such that \( s(t_0) \neq s(t_1) \) for all \( 1 \leq i \leq 7 \) and \( s(t_i) \) for \( i \) even \( \neq s(t_j) \) for \( i \) odd.

Graph admits 2-divisor lucky labeling and to label this graph only two labels has been used. Therefore 2-divisor lucky number is two i.e \( \eta_{vdl}(G) = 2 \).

### 4. [7] [9] 2-DIVISOR LUCKY LABELING FOR SOME ZERO-DIVISOR GRAPHS

**Theorem 4.1:** 2-divisor lucky number is one for the zero-divisor graph \( G \).

**Proof:** Let graph \( G = \Gamma(Z_{2p}) \) and \( p > 3 \) be a prime number.

\[
V(G) = \{t_1, t_2, ..., t_{p-1}, t_p\} = \{2,4,...,2(p-1), p\}, E(G) = \{t_it_{p-1}/1 \leq i \leq p - 1\}.
\]

\[
|V(G)| = p ; |E(G)| = p - 1. \] Define \( g: V(G) \rightarrow \{1,2,3,...,p\} \) such that \( f(t_i) = 1, 1 \leq i \leq p \).

\[
s(t_p) = \left[ \frac{g(t_1) + g(t_2) + ... + g(t_{p-1})}{p} \right]
\]

\[
s(t_i) = \left[ \frac{g(t_p)}{2} \right] 1 \leq i \leq p - 1
\]

such that \( s(t_p) \neq s(t_i) \) for all \( 1 \leq i \leq p - 1 \). Only one label has been used to label this graph \( G \). Therefore 2-divisor lucky number is one i.e \( \eta_{vdl}(G) = 1 \).

**Theorem 4.2** The 2-divisor lucky number for the zero-divisor graph \( \Gamma(Z_{3p}), p > 3 \) is one.

**Proof:** Let graph \( G = \Gamma(Z_{2p}) \) and \( p > 3 \), be a prime number.

\[
V(G) = \{s_1, s_2, t_1, t_2, t_3, ..., t_{p-1}\} = \{p, 2p, 3, 5, 6, 9, ..., 3(p-1)\}.
\]

\[
E(G) = \{s_it_j/1 \leq i \leq 2, 1 \leq j \leq p - 1\}. \] Define \( g: V(G) \rightarrow \{1,2,3,...,2p - 2\} \) such that \( f(s_i) = 1, 1 \leq i \leq 2 \) and \( f(t_j) = 1, 1 \leq j \leq p - 1 \)

\[
s(s_i) = \left[ \frac{g(t_1) + g(t_2) + ... + g(t_{p-1})}{2p-2} \right] \text{ where } i = 1,2
\]
Theorem: 4.3 $\Gamma(Z_{5p}), p \geq 3, \text{ and } p \neq 5$ the zero-divisor has 2-divisor lucky number to be one.

Proof: Let graph $G = \Gamma(Z_{5p}), p \geq 3$ and $p \neq 5$ be a prime number.

$V(G) = \{s_1, s_2, s_3, t_1, t_2, t_3, \ldots, t_{p-1}\} = \{p, 2p, 3p, 4p, 5, 10, 15, \ldots, 5(p - 1)\}$

$E(G) = \{s_it_j/1 \leq i \leq 4, 1 \leq j \leq p - 1\}. |V(G)| = p + 3$ ; $|E(G)| = 4p - 4$.

Define $g: V(G) \rightarrow \{1,2,4p - 4\}$ such that $g(s_i) = 1, 1 \leq i \leq 4$ and $g(t_j) = 1, 1 \leq j \leq p - 1$

$s(s_i) = \left\lfloor \frac{g(s_1) + g(s_2) + \ldots + g(s_4)}{2} \right\rfloor$ where $1 \leq i \leq 4$

$s(t_j) = \left\lfloor \frac{g(t_1) + g(t_2) + \ldots + g(t_{p-1})}{2} \right\rfloor$ where $1 \leq j \leq p - 1$

such that $s(s_i) \neq s(t_j)$ for all $1 \leq i \leq 4$ and $1 \leq j \leq p - 1$.

Therefore the 2-divisor lucky number for graph $G$ is one. i.e. $\eta_{vd}(G) = 1$.

Theorem: 4.4 The 2-divisor lucky number, for the zero-divisor graph $\Gamma(Z_2 \times Z_p), p \geq 3$, is one.

Proof: Let graph $G = \Gamma(Z_2 \times Z_p), p \geq 3$ be a prime number.

$V(G) = \{t_1, t_2, t_3, \ldots, t_{p-1}, x\} = \{(0,1), (0,2), (0,3), \ldots, (0,p - 1), (1,0)\}$

$E(G) = \{t_i x/1 \leq i \leq p - 1\}. |V(G)| = p$ ; $|E(G)| = p - 1$.

Define $g: V(G) \rightarrow \{1,2,3,\ldots,p\}$ such that $g(t_i) = 1, 1 \leq i \leq p - 1$ and $g(x) = 1$.

$s(t_i) = \left\lfloor \frac{g(t_i)}{2} \right\rfloor$ for all $1 \leq i \leq p - 1$

$s(x) = \left\lfloor \frac{g(t_1) + g(t_2) + \ldots + g(t_{p-1})}{2} \right\rfloor$

such that $s(t_i) \neq s(x)$ for all $1 \leq i \leq p - 1$.

Graph $G$ admits 2-divisor lucky labeling and the 2-divisor lucky number of graph $G$ is one. i.e. $\eta_{vd}(G) = 1$.

Theorem: 4.5 The 2-divisor lucky number for the zero-divisor graphs, be a prime number is three.

Proof: Let graph $G = \Gamma(Z_{2p}) + \Gamma(Z_4), p \geq 3$ be a prime number.

$V(G) = \{t_1, t_2, t_3, \ldots, t_{p-1}, t_p, x\} = \{2,4,6, \ldots, 2(p - 1), p, 2\}$ where $2 \in Z_4$.

$E(G) = \{t_i x, t_p t_j, t_{p}x/1 \leq i \leq p - 1\}. |V(G)| = p + 1$ ; $|E(G)| = 2p - 1$. 

SRMS Journal of Mathematical Sciences, Vol-6, 2020, pp. 1-7 ISSN: 2394-725X 5
Define \( g: V(G) \rightarrow \{1, 2, 3, \ldots, 2p - 1\} \) such that 
\[
g(t_i) = 2, \quad 1 \leq i \leq p - 1, \quad g(t_p) = 1 \quad \text{and} \quad g(x) = 3.
\]

\[
s(t_p) = \left\lfloor \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(x)}{2} \right\rfloor
\]

\[
s(t_i) = \left\lfloor \frac{g(t_p) + g(x)}{2} \right\rfloor \quad \text{for all} \quad 1 \leq i \leq p - 1
\]

\[
s(x) = \left\lfloor \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p)}{2} \right\rfloor \quad \text{such that} \quad s(t_i) \neq s(x) \neq s(t_p) \quad \text{for all} \quad 1 \leq i \leq p - 1.
\]

The 2-divisor lucky number of graph \( G \) is three i.e. \( \eta_{vl}(G) = 3 \).

**Theorem: 4.6** 2-divisor lucky number is three for the zero-divisor graphs \( \Gamma(Z_{2p}) + \Gamma(Z_6), \ p > 3 \).

**Proof:** Let graph \( G = \Gamma(Z_{2p}) + \Gamma(Z_6), \ p \geq 3 \) be a prime number.

\( V(G) = \{t_1, t_2, t_3, \ldots, t_{p-1}, t_p, s, t, u\} = \{2, 4, 6, \ldots, 2(p - 1), p, 2, 3, 4\} \) where \( 2, 3, 4 \in Z_6 \).

\( E(G) = \{t_is, t_it, t_itu, t_pti, t_ps, t_pt, t_ps, s, tu, st, tu/1 \leq i \leq p - 1\}. |V(G)| = p + 3; \ |E(G)| = 4p + 1. \)

Define \( g: V(G) \rightarrow \{1, 2, 3, \ldots, p + 3\} \) such that 
\[
g(x_i) = 3, \quad 1 \leq i \leq p - 1, \quad g(x_p) = 1, \quad g(s) = 2, \quad g(t) = 3, \quad g(u) = 2
\]

\[
s(t_p) = \left\lfloor \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(s) + g(t) + g(u)}{2} \right\rfloor
\]

\[
s(t_i) = \left\lfloor \frac{g(t_p) + g(x) + g(t) + g(u)}{2} \right\rfloor \quad \text{for all} \quad 1 \leq i \leq p - 1
\]

\[
s(s) = \left\lfloor \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(t)}{2} \right\rfloor
\]

\[
s(t) = \left\lfloor \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(s) + g(x)}{2} \right\rfloor
\]

\[
s(u) = \left\lfloor \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(t)}{2} \right\rfloor \quad \text{such that} \quad s(t_i) \neq s(s) \neq s(t) \neq s(u) \neq s(t_p)
\]

for all \( 1 \leq i \leq p - 1. \) Therefore the 2-divisor lucky number of graph \( G \) is three i.e. \( \eta_{vl}(G) = 3 \).

**Theorem: 4.7** 2 is the 2-divisor lucky number for the zero-divisor graphs \( \Gamma(Z_{2p}) + \Gamma(Z_9), \ p > 3. \)

**Proof:** Let graph \( G = \Gamma(Z_{2p}) + \Gamma(Z_9), \ p > 3 \) be a prime number.

\( V(G) = \{t_1, t_2, t_3, \ldots, t_{p-1}, t_p, s, t\} = \{2, 4, 6, \ldots, 2(p - 1), p, 3, 6\} \) where \( 3, 6 \in Z_9. \)
K. Aruna Sakthi et al.

\[ E(G) = \{ t_1s, t_1t_2, t_2s, t_2t_3, \ldots, t_{p-1}s, t_{p-1}t \} \quad \text{with} \quad |V(G)| = p + 2 \quad \text{and} \quad |E(G)| = 3p. \]

Define \( g: V(G) \to \{1, 2, 3, \ldots, p+2\} \) such that \( g(t_i) = 2, 1 \leq i \leq p \), \( g(s) = 1, g(t) = 1 \).

\[
s(t_p) = \left[ \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(s) + g(t)}{2} \right]
\]

\[
s(t_i) = \left[ \frac{g(t_p) + g(s) + g(t)}{2} \right] \quad \text{for all} \quad 1 \leq i \leq p - 1.
\]

\[
s(s) = \left[ \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(t)}{2} \right]
\]

\[
s(t) = \left[ \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(s)}{2} \right]; \quad \text{such that} \quad s(t_i) \neq s(t) \neq s(s) \neq s(t_p) \quad \text{for all} \quad 1 \leq i \leq p - 1.
\]

Therefore \( \eta_{vdt}(G) = 2 \), 2-divisor lucky number is two.

Reference


