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2-Divisor Lucky Labeling of some Identity Graphs of Finite Group and some Zero-Divisor Graphs

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Abstract

A graph $G = (V, E)$ be a graph with n vertices and m edges. A graph G admits 2-divisor lucky labeling if $f: V(G) \to \{1,2,\dots,n\}$ be a labeling of vertices of graph G from {1,2,3,…,n}. Define $s(u) = \lceil \sum_{v \in N(v)} \frac{f(v)}{2} \rceil$, where N(v) is the neighborhood of v such that s(u) ≠ s(v) for every pair of adjacent vertices u and v in G. The vertex 2-divisor lucky number is the least number from the set {1,2,…,n} that has been used to label the graph G. It is denoted by η_{vdl}. In this paper we have investigated for some types of identity graphs of finite group and some zero-divisor graphs.

1. Introduction

The zero-divisor graph has two types. One is in the Beck definition (1988) in which all the elements in the rings will be consider for the vertex set[4]. Other is in the Anderson and Livingston[3] definition he slightly varied by considering only zero-divisor for the vertex set in the year 1999. Identity graphs, semigroups and some special subgraphs was studied by Kandasamy and Smarandache in this paper[6]. Rosa[8] was the one who introduced graph labeling in the year 1967. Labeling[5] has many application in the field of Engineering and technology etc. Lucky labeling was studied by Ahai.et.al and Akbari.et.al[1][2]. Applications of lucky labeling is in transportation network, to model protein structure etc. Inspiring d-lucky labeling and various type of lucky labeling 2-divisor lucky labeling has been introduce in this paper and studied for some identity graphs of finite groups and some zero-divisor graphs[9][7].

2. Preliminaries

Definition 2.1: Zero-Divisor Graph:[7]

Let R be a commutative ring with identity 1 and let $Z(R)$ be its set of zero-divisors. We associate a $\Gamma(R)$ to R with vertices, $Z^* = Z(R) - \{0\}$, the set nonzero zero-divisor of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. We denote their zero-divisor graph of R by $\Gamma_0(R)$ if we take vertex set as Z (R). In $\Gamma_0(R)$, the vertex 0 is adjacent to every other vertex. $\Gamma(R)$ is a induced subgraph of $\Gamma_0(R)$.

Definition 2.2: Identity Graph:

Let g be a group. The identity graph $G = (V, E)$ with vertices as the elements of group and two elements $x, y \in g$ are adjacent or can be joined by an edge if $x.y = e$, where e is the identity element of g and identity element is adjacent to every other vertices in G.

3. 2-DIVISOR LUCKY LABELING FOR SOME IDENTITY GRAPHS

Definition: 2-Divisor Lucky Labeling:

A graph $G = (V,E)$ be a graph with n vertices and m edges. A graph G admits 2-divisor lucky labeling if f: V(G)→{1,2,…,n} be a labeling of vertices of graph G from {1,2,3,…,n}.

Define, $s(u) = \lceil \sum_{v \in N(v)} \frac{f(v)}{2} \rceil$, where $N(v)$ is the neighborhood of v such that $s(u) \neq s(v)$ for every pair of adjacent vertices u and v in G The vertex 2-divisor lucky number is the least number from the set {1,2,…,n} that has been used to label the graph G. It is denoted by η_{vdl}.

Theorem: 3.1 2-divisor lucky number for the identity graph of for be an odd number is two.

Proof: Let G = graph *of* Z_n, n > 3 Identity graph be an odd number.

$$V(G) = \{0,1,2, \dots n-1\} = \left\{t_0, t_1, t_2, \dots, t_{\frac{n-1}{2}}, t_{\frac{n+1}{2}}, \dots, t_{n-1}\right\}.$$

$E(G) = \{t_0 t_i,\ t_1 t_{n-1}, t_2 t_{n-2}, \dots, t_{\frac{n-1}{2}}\ t_{\frac{n+1}{2}}\},\ 1 \leq i \leq n-1.\ \ |V(G)| = n\ ; |E(G)| = \frac{3n-3}{2}$

Define $g: V(G) \to \{1,2, \dots, n\}$ such that $g(t_0) = 1$, $g(t_1) = g(t_2) = \cdots = g\left(t_{\frac{n-1}{2}}\right) = 1$ and

$g(t_{\frac{n+1}{2}}) = g\left(t_{\frac{n+3}{2}}\right) = \cdots = g(t_{n-1}) = 2.$

$$s(t_0) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{n-1})}{2} \right\rceil$$

$$s(t_1) = \left\lceil \frac{g(t_0) + g(t_{n-1})}{2} \right\rceil$$

$$s(t_2) = \left\lceil \frac{g(t_0) + g(t_{n-2})}{2} \right\rceil$$

$$s\left(t_{\frac{n-1}{2}}\right) = \left\lceil \frac{g(t_0) + g\left(t_{\frac{n+1}{2}}\right)}{2} \right\rceil$$

$$s(t_{n-1}) = \left\lceil \frac{g(t_0) + g(t_1)}{2} \right\rceil$$

such that $s(t_0)\ \neq s(t_1) \neq\ s(t_2) \neq \cdots \neq\ s\left(t_{\frac{n-1}{2}}\right) \neq s(t_{n-1})$.

To label this Graph only two labels has been used and also admits 2-divisor lucky labeling. Therefore 2-divisor lucky number is two i.e η_{vdl} *(G)=2.*

Theorem: 3.2 The identity graph of $(Z_n, \oplus_n)$ for *n > 2* be an even number has 2-divisor lucky number to be two.

Proof: Let graph *G = Identity graph of* Z_n, $n > 2$ be an even number.

$$V(G) = \{0,1,2, \dots n-1\} = \left\{t_0, t_1, t_2, \dots, t_{\frac{n}{2}}, \dots, t_{n-1}\right\}.$$

$E(G) = \{t_0 t_i,\ t_1 t_{n-1}, t_2 t_{n-2}, \dots, /\ \ 1 \leq i \leq n-1\}.\ |V(G)| = n\ ; |E(G)| = \frac{3n-3}{2}$

Define $g: V(G) \to \{1,2, \dots, n\}$ such that $g(x_0) = 1$, and $g(x_i) = \begin{cases} 1\ if\ n = odd \\ 2\ \ if\ n = even \end{cases}$

$1 \leq i \leq n-1\ and\ i \neq \frac{n}{2}\quad g\left(x_{\frac{n}{2}}\right) = 2.$

$$s(t_0) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{n-1})}{2} \right\rceil$$

$$s(t_1) = \left\lceil \frac{g(t_0) + g(t_{n-1})}{2} \right\rceil$$

$$s(t_2) = \left\lceil \frac{g(t_0) + g(t_{n-2})}{2} \right\rceil$$

$$s\left(t_{\frac{n}{2}}\right) = \left\lceil \frac{g(t_0)}{2} \right\rceil$$

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$$s(t_{n-1}) = \left\lceil \frac{g(t_0) + g(t_1)}{2} \right\rceil$$

such that $s(t_0) \neq s(t_1) \neq s(t_2) \neq \cdots \neq s\left(t_{\frac{n-1}{2}}\right) \neq s(t_{n-1})$.

Graph *G* admits 2-divisor lucky labeling and to label this graph only two labels has been used. Therefore 2-divisor lucky number is two i.e $\eta_{vdl}(G)=2$.

Theorem: 3.3 Identity graph of Klein-4 group under composition has to be one.

Proof: Let graph *G* = Identity graph of Klein-4 group

$V(G) = \{t_0, t_1, t_2, t_3\} = \{e, a, b, ab\}$. $E(G) = \{t_0 t_i / 1 \leq i \leq 3\}$.

Define $g: V(G) \rightarrow \{1,2,3,4\}$ such that $g(t_i) = 1$ for all $1 \leq i \leq 4$.

$$s(t_0) = \left\lceil \frac{g(t_1) + g(t_2) + g(t_3) + g(t_4)}{2} \right\rceil$$

$s(t_i) = \left\lceil \frac{g(t_0)}{2} \right\rceil$ for all $1 \leq i \leq 3$.

such that $s(t_0) \neq s(t_i)$ for all $1 \leq i \leq 3$.

Only one label has been used to label this graph *G*. Therefore $\boldsymbol{\eta_{vdl}}$ **(G)** is **1.**

Theorem: 3.4 The 2-divisor lucky number for the identity graph of quaternion group is two.

Proof: Let graph *G* = identity graph of Q_8.

$V(G) = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$.

$E(G) = \{t_0 t_i, t_2 t_3, t_4 t_5, t_6 t_7 : 1 \leq i \leq 7\}$.

Define $g: V(G) \to \{1,2,3 \ldots ,8\}$ such that, $g(t_0) = 1$, $g(t_i) = \begin{cases} 2 \; if \; n \; is \; even \\ 1 \; if \; n \; is \; odd \end{cases}$; $1 \le i \le 7$.

$$s(t_0) = \left\lceil \frac{g(t_1) + g(t_2) + g(t_3) + g(t_4) + g(t_5) + g(t_6) + g(t_7)}{2} \right\rceil$$

$$s(t_1) = \left\lceil \frac{g(t_0)}{2} \right\rceil$$

$s(t_i) = \left\lceil \frac{g(t_0)+g(t_{i+1})}{2} \right\rceil$ if $i = even$

$s(t_i) = \left\lceil \frac{g(t_0)+g(t_{i+1})}{2} \right\rceil$ if i is odd except $i = 1$

such that $s(t_0) \neq s(t_i)$for all $1 \le i \le 7$ and $s(t_i) for \; i \; even \neq s(t_j) for \; i \; odd.$

Graph admits 2-divisor lucky labeling and to label this graph only two labels has been used. Therefore 2-divisor lucky number is two i.e η_{vdl} (G) = 2.

4. [7] [9] 2-DIVISOR LUCKY LABELING FOR SOME ZERO-DIVISOR GRAPHS

Theorem 4.1: 2-divisor lucky number is one for the zero-divisor graph, .

Proof: Let graph $G = \Gamma(Z_{2p})$ and $p > 3$ be a prime number.

$V(G) = \{t_1, t_2, \ldots, t_{p-1}, t_p\} = \{2,4, \ldots ,2(p-1), p\}$, $E(G) = \{t_i t_p / 1 \le i \le p-1\}$.

$|V(G)| = p$; $|E(G)| = p - 1$. Define $g: V(G) \to \{1,2,3, \ldots, p\}$ such that $f(t_i) = 1$,

$1 \le i \le p$.

$$s(t_p) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1})}{2} \right\rceil$$

$$s(t_i) = \left\lceil \frac{g(t_p)}{2} \right\rceil \; 1 \le i \le p-1$$

such that $s(t_p) \neq s(t_i)$ for all $1 \le i \le p-1$. Only one label has been used to label this graph G. Therefore 2-divisor lucky number is one i.e $\boldsymbol{\eta_{vdl}(G) = 1}$.

Theorem: 4.2 The 2-divisor lucky number for the zero-divisor graph $\Gamma(Z_{3p})$, p > 3 is one.

Proof: Let graph G= $\Gamma(Z_{3p})$ and $p > 3$, be a prime number.

$V(G) = \{s_1, s_2, t_1, t_2, t_3, \ldots, t_{p-1}\} = \{p, 2p, 3,6,9, \ldots ,3(p-1)\}$.

$E(G) = \{s_i t_j / 1 \le i \le 2, 1 \le j \le p-1\}$. $|V(G)| = p + 1$; $|E(G)| = 2p - 2$.

Define $g: V(G) \to \{1,2,3, \ldots ,2p-2\}$ such that $f(s_i) = 1$, $1 \le i \le 2$ and $f(t_j) = 1$,

$1 \le j \le p-1$

$s(s_i) = \left\lceil \frac{g(t_1)+g(t_2)+\cdots+g(t_{p-1})}{2} \right\rceil$ where $i = 1,2$

$s(t_j) = \left\lceil \frac{g(s_1)+g(s_2)}{2} \right\rceil$ where $1 \leq j \leq p-1$

such that $s(s_i) \neq s(t_j)$ for all $1 \leq i \leq 2$ and$1 \leq j \leq p-1$.

Labeling this graph only one label has been used so 2-divisor lucky number is two. i.e.
$\boldsymbol{\eta_{vdl}(G) = 1}$.

Theorem: 4.3 $\Gamma(Z_{5p}), p \geq 3$, and $p \neq 5$ the zero-divisor has 2-divisor lucky number to be one.

Proof: Let graph $G = \Gamma(Z_{5p}), p \geq 3$ and $p \neq 5$ be a prime number.

$V(G) = \{s_1, s_2, s_3, s_4, t_1, t_2, t_3, \ldots, t_{p-1}\} = \{p, 2p, 3p, 4p, 5, 10, 15, \ldots, 5(p-1)\}$

$E(G) = \{s_i t_j / 1 \leq i \leq 4, 1 \leq j \leq p-1\}$. $|V(G)| = p+3$; $|E(G)| = 4p-4$.

Define $g: V(G) \to \{1,2,\ldots,4p-4\}$ such that $g(s_i) = 1, 1 \leq i \leq 4$ and $g(t_j) = 1, 1 \leq j \leq p-1$

$s(s_i) = \left\lceil \frac{g(t_1)+g(t_2)+\cdots+g(t_{p-1})}{2} \right\rceil$ where $1 \leq i \leq 4$

$s(t_j) = \left\lceil \frac{g(s_1)+g(s_2)+g(s_3)+g(s_4)}{2} \right\rceil$ where $1 \leq j \leq p-1$

such that $s(s_i) \neq s(t_j)$ for all $1 \leq i \leq 4$ and $1 \leq j \leq p-1$.

Therefore the 2-divisor lucky number for graph G is one. i.e. $\boldsymbol{\eta_{vdl}(G) = 1}$.

Theorem: 4.4 The 2-divisor lucky number, for the zero-divisor graph $\Gamma(Z_2 \times Z_p)$, $p \geq 3$, is one.

Proof: Let graph $G = \Gamma(Z_2 \times Z_p), p \geq 3$ be a prime number.

$V(G) = \{t_1, t_2, t_3, \ldots, t_{p-1}, x\} = \{(0,1), (0,2), (0,3), \ldots, (0, p-1), (1,0)\}$

$E(G) = \{t_i x / 1 \leq i \leq p-1\}$. $|V(G)| = p$; $|E(G)| = p-1$.

Define $g: V(G) \to \{1,2,3,\ldots,p\}$ such that $g(t_i) = 1, 1 \leq i \leq p-1$ and $g(x) = 1$.

$s(t_i) = \left\lceil \frac{g(x)}{2} \right\rceil$ for all $1 \leq i \leq p-1$

$$s(x) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1})}{2} \right\rceil$$

such that $s(t_i) \neq s(x)$ for all $1 \leq i \leq p-1$.

Graph G admits 2-divisor lucky labeling and the 2-divisor lucky number of graph G is onei.e.
$\boldsymbol{\eta_{vdl}(G) = 1}$.

Theorem: 4.5 The 2-divisor lucky number for the zero-divisor graphs , be a prime number is three.

Proof: Let graph $G = \Gamma(Z_{2p}) + \Gamma(Z_4), p \geq 3$ be a prime number.

$V(G) = \{t_1, t_2, t_3, \ldots, t_{p-1}, t_p, x\} = \{2,4,6, \ldots, 2(p-1), p, 2\}$ where $2 \in Z_4$.

$E(G) = \{t_i x,\ t_p t_i,\ t_p x / 1 \leq i \leq p-1\}$. $|V(G)| = p+1$; $|E(G)| = 2p-1$.

Define $g: V(G) \rightarrow \{1,2,3..,2p-1\}$ such that $g(t_i) = 2$, $1 \leq i \leq p-1$, $g(t_p) = 1$ and $g(x) = 3$.

$$s(t_p) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(x)}{2} \right\rceil$$

$s(t_i) = \left\lceil \frac{g(t_p)+g(x)}{2} \right\rceil$ for all $1 \leq i \leq p-1$

$s(x) = \left\lceil \frac{g(t_1)+g(t_2)+\cdots+g(t_{p-1})+g(t_p)}{2} \right\rceil$ such that $s(t_i) \neq s(x) \neq s(t_p)$ for all $1 \leq i \leq p-1$.

The 2-divisor lucky number of graph G is three i.e. $\boldsymbol{\eta_{vdl}(G) = 3}$.

Theorem: 4.6 2-divisor lucky number is three for the zero-divisor graphs $\Gamma(Z_{2p}) + \Gamma(Z_6)$, $p > 3$.

Proof: Let graph $G = \Gamma(Z_{2p}) + \Gamma(Z_6)$, $p \geq 3$ be a prime number.

$V(G) = \{t_1, t_2, t_3, \dots, t_{p-1}, t_p, s, t, u\} = \{2,4,6, \dots, 2(p-1), p, 2,3,4\}$ where $2,3,4 \in Z_6$.

$E(G) = \{t_i s,\ t_i t,\ t_i u,\ t_p t_i,\ t_p s, t_p t,\ t_p u, st, tu / 1 \leq i \leq p-1\}$. $|V(G)| = p+3$; $|E(G)| = 4p+1$.

Define $g: V(G) \rightarrow \{1,2,3 \dots, p+3\}$ such that $g(x_i) = 3$ $1 \leq i \leq p-1$, $g(x_p) = 1$,

$g(s) = 2$, $g(t) = 3$, $g(u) = 2$

$$s(t_p) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(s) + g(t) + g(u)}{2} \right\rceil$$

$s(t_i) = \left\lceil \frac{g(t_p)+g(s)+g(t)+g(u)}{2} \right\rceil$ for all $1 \leq i \leq p-1$

$$s(s) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(t)}{2} \right\rceil$$

$$s(t) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(s) + g(u)}{2} \right\rceil$$

$s(u) = \left\lceil \frac{g(t_1)+g(t_2)+\cdots+g(t_{p-1})+g(t_p)+g(t)}{2} \right\rceil$; such that $s(t_i) \neq s(s) \neq s(t) \neq s(u) \neq s(t_p)$ for all $1 \leq i \leq p-1$. Therefore the 2-divisor lucky number of graph G is three i.e. $\boldsymbol{\eta_{vdl}(G) = 3}$.

Theorem: 4.7 2 is the 2-divisor lucky number for the zero-divisor graphs $\Gamma(Z_{2p}) + \Gamma(Z_9)$, $p > 3$.

Proof: Let graph $G = \Gamma(Z_{2p}) + \Gamma(Z_9)$, $p > 3$ be a prime number.

$V(G) = \{t_1, t_2, t_3, \dots, t_{p-1}, t_p, s, t\} = \{2,4,6, \dots, 2(p-1), p, 3,6\}$ where $3,6 \in Z_9$.

$E(G) = \{t_i s, t_i t \; t_p t_i, \; t_p s, t_p t, st / 1 \le i \le p-1\}$. $|V(G)| = p+2$; $|E(G)| = 3p$.

Define $g: V(G) \to \{1,2,3 \ldots, p+2\}$ such that $g(t_i) = 2 \;\; 1 \le i \le p$, $g(s) = 1, g(t) = 1$.

$$s(t_p) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(s) + g(t)}{2} \right\rceil$$

$s(t_i) = \left\lceil \frac{g(t_p)+g(s)+g(t)}{2} \right\rceil$ for all $1 \le i \le p-1$

$$s(s) = \left\lceil \frac{g(t_1) + g(t_2) + \cdots + g(t_{p-1}) + g(t_p) + g(t)}{2} \right\rceil$$

$s(t) = \left\lceil \frac{g(t_1)+g(t_2)+\cdots+g(t_{p-1})+g(t_p)+g(s)}{2} \right\rceil$; such that $s(t_i) \ne s(t) \ne s(s) \ne s(t_p)$ for all $1 \le i \le p-1$. Therefore $\boldsymbol{\eta_{vdl}(G) = 2}$, 2-divisor lucky number is two.

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