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Mathematical Model to Study the Heat Transfer between Core and Skin

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Heat Flow, Perfusion Rate, Metabolic Heat Generation, Skin, Tissue, Temperature.

Abstract

A one-dimensional analytical model for heat transfer between core and skin is developed, and the relation between heat and distance from the core is obtained. This paper is a generalization of the earlier works by incorporating the effects of temperature on arterial and venousbeds andmetabolic heat generation depending on temperature and different atmospheric conditions. In this work it is considered that the epidermis layer is negligibly thin for mathematical simplification. The results are obtained by taking variations in the coefficient of heat metabolism and metabolic heat generalization. In this work it is obtained the heat in tissue increases from core to surface in both cases where the coefficient of metabolism and metabolic heat generation are varied.

Introdution

The skin is primarily divided into three levels called the epidermis, dermis, and hypo-dermis. The upper layer epidermisconsists of pores for perspiration and hairs [9]. This paper considers mainly the second layer dermis as it is the region where blood vessels and the capillary bed are present. It is the region where heat transfer takes place. The bottom-most layer is highly dense, and it is the layer that connects skin to muscles and bones [8].The energy is supplied to tissue in various ways, such as energy provided by blood flow, energy generated by the metabolic process, heat transport by a temperature gradient, and energy transferred to the environment by conduction, convection, radiation, and evaporation. The oxygenated blood coming through arteries carries heat. The blood flows through the capillary bed to provide oxygen and nutrients to cells present in the tissue. In that process, the heat carried by the artery is given to tissue [20, 22]. When deoxygenated blood moves through the capillary bed, it also does the same. In paper [13], they have developed a new model for heat transfer between an ambient atmosphere and isothermal core. The outermost thickness of the skin was considered to be very small, and calculations were made in the absence of metabolic heat transfer between tissues and vein or arteries. The human body needs a fixed temperature, i.e., 37 ^{0}C, for its normal functioning. Therefore the core has to be isothermal [5, 12]. An increase or decrease in the core temperature affects the normal functioning of organs. It has been discovered that the body has controls available to it for the maintenance of a homo-isothermal state by varying its resistance to heat flow. A definite vascular structure has been identified, which appears to function toprotect heat. In these structures, arteries and veins are arranged closely packed for counter-current flow [4, 10]. A system regulates the balance between heat lost in the body and heat produced to the environment to sustain a constant core temperature. Temperature regulating center in the hypothalamus is responsive tothe temperature of blood circulation system. This center react to temperature change by forwarding nerve impulses to the dermis arterioles. When body temperature rises, vasodilation happens, and core temperature comes down. When body temperature falls, vasoconstriction occurs, and heat is conserved. It was also found that the rate of loss of heat from an extremity is related to local blood perfusion rate [16, 21]. In paper [18, 19] the authors have done significant work in this field. He imported previous models by treating tissue as a continuum of finite thickness and developed appropriate heat conservation differential equations to describe the tissue profile. Wessler [24] solved more complexreal world problem of the heat balance by dividing the body into a small number of cylindrical regions thermally attached by the circulatory flow. His model also took conductive and convective effects, but it has been assumed that arterial and venous blood temperatures are uniform in each region so that the effect of local counter-current heat exchange is lost [14, 15]. Gupta and Tandon [23] had developed a new model for transfer of heat between core and skin, which incorporated the effects of local variable perfusion rate in addition to convective heat transfer and metabolic heat generation depending on temperature [6, 17]. Recently, research is being focussed on Hilfer fractional equation method to solve differential equations [1]. This method has the advantage of predicting existence and controllability of the solutions obtained [3, 7]. Another technique for stability analysis is Langevin Fractional equation method. The Caputo operator is used to obtain a fractional version of the vector-borne model. Using the Banach contraction principle theorem, the existence and uniqueness analysis of the mathematical models can also be established [2, 3]. But this paper deals with linear differential equation models only.In earlier models, an attempt was made for either no metabolic heat generation rate or constant arterial and venous temperature. This paper is a further generalization of the earlier work by incorporating the effects of temperature on arterial and venous beds and metabolic heat generation depending on temperature and different atmospheric conditions.

Formulation of the Problem

The schematic picture of the present model hasdescribed in the given Fig. 1. The assumed thickness of peripheral layer is very small compared to the radius of curvature of the local surface. Since the region is very small, therefore, we can consider the temperature distribution only in the direction normal to the surface. Hence the system can be considered to be one-dimensional. The steady-state heat balance equations can be written as follows,

$$k\frac{d^2T}{dx^2} + (h.a + C_p.g)(T_a - T) + h.a(T_v - T) + M = 0 \quad (1)$$

$$\left[\overset{0}{m}_a - \int_0^x g\,dx\right] C_p \frac{dT_a}{dx} + h.a(T_a - T) = 0 \quad (2)$$

$$C_p\left[\overset{0}{m}_v + \int_0^x g\,dx\right]\frac{dTv}{dx} + (h.a + C_p.g)(T_v - T) = 0 \quad (3)$$

Where K is the thermal conductivity of the tissue, h is heat transfer coefficient, a is area of cross section, C_p is the specific heat capacity, g is the tissue perfusion rate, T_a is the temperature of the artery, T_v is the temperature of the vein, M is the metabolic heat generation rate, $\dot{m}_a$is the arterial flow rate, $\dot{m}_v$ is the venous flow rate.

Assuming M is varying linearly with T, we get the following relation,

$$M = M_0(1 + \alpha T) \quad (4)$$

Where M_0 is the thermal energy generation for resting muscle and skin [15], and α is the metabolic factor or coefficient of metabolism.

Solution of the Problem

Considering h.a = 0, g = constant, $M = M_0(1 + \alpha T)$ the above equations 1,2, and 3 reduce as follows. The value h.a is considered to be zero because there is no vascular interaction i.e., no heat exchange occurs between artery and tissue or veins and tissue [11].

$$K\frac{d^2T}{dx^2} + (M_0\alpha - C_p g)T + C_p g T_a + M_0 = 0 \quad (5)$$

$$\frac{dT_a}{dx} = 0 \qquad \text{i.e. } T_a = \text{constant} \quad (6)$$

$$\left(\overset{0}{m}_v + gx\right)\frac{dT_v}{dx} + g(T_v - T) = 0 \quad (7)$$

The boundary conditions for governing equations are given as follows,

$At \quad x = 0, \quad T = T_a \quad = T_c$

$At \quad x = L, \quad T = T_v \quad = T_s$

$$T = C_1\cos\left(\sqrt{\lambda_1 x}\right) + C_2\sin\left(\sqrt{\lambda_1 x}\right) - \frac{\lambda_2}{\lambda_1} \quad (8)$$

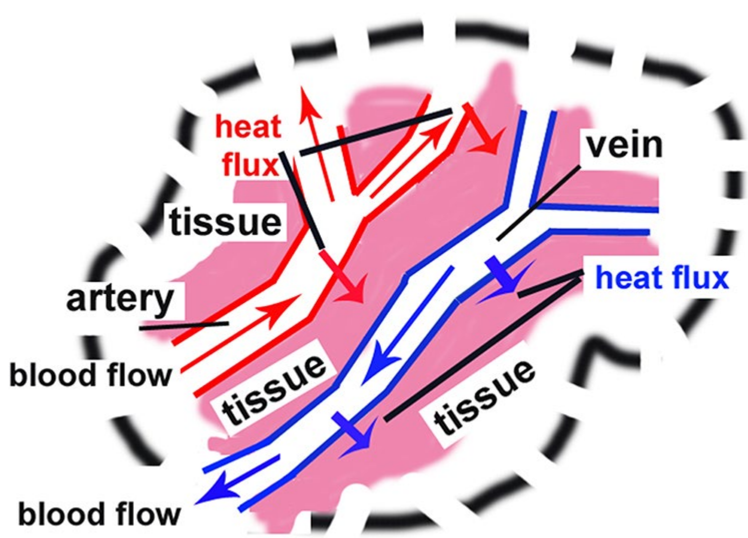

Fig. 1: Schematic Diagram of Tissue Region Emphasizing its Vascular and Temperature Variations.

where,

$$C_1 = T_c + \frac{\lambda_2}{\lambda_1} \tag{9}$$

$$C_2 = \frac{T_s - \left[T_c + \frac{\lambda_2}{\lambda_1}\right]\cos\left(\sqrt{\lambda_1}.L\right) + \frac{\lambda_2}{\lambda_1}}{\sin\left(\sqrt{\lambda_1}L\right)} \tag{10}$$

$$\lambda_1 = \frac{M_0\alpha - C_p g}{K} \tag{11}$$

$$\lambda_2 = \frac{C_p.g.T_a + M_0}{K} \tag{12}$$

Substituting the above value of T from Equation (7) into Equation (6) and solving for T_v, T_v, we get

$$T_v = \frac{g}{\overset{0}{m}_v + gx}\left[\frac{C_1\sin\sqrt{\lambda_1}x}{\lambda_1} - \frac{C_2\cos\sqrt{\lambda_1}x}{\lambda_1} - \frac{\lambda_2}{\lambda_1}x\right] + C_4 \tag{13}$$

Where,

$$C_4 = T_s - \frac{g}{\overset{0}{m}_v + gL}\left[\frac{C_1\sin\sqrt{\lambda_1}L}{\sqrt{\lambda_1}} - \frac{C_2\cos\sqrt{\lambda_1}L}{\sqrt{\lambda_1}} - \frac{\lambda_2}{\lambda_1}L\right] \tag{14}$$

Now, using the followingDimensionless variables,

$$\theta_1 = \frac{T - T_c}{T_s - T_c}, \quad \theta_2 = \frac{T_v - T_c}{T_s - T_c}, \quad X = \frac{x}{L} \tag{15}$$

$$\theta_1 = \left(\frac{C_1}{T_s - T_c}\right)\cos\left(\sqrt{\lambda_1}LX\right) + \left(\frac{C_2}{T_s - T_c}\right)\sin\left(\sqrt{\lambda_1}LX\right) - \frac{\lambda_2}{\lambda_1(T_s - T_c)} - \frac{T_c}{T_s - T_c} \tag{16}$$

$$\theta_2 = \frac{g}{\left(\overset{0}{m}_v + g.L.X\right)(T_s - T_c)}\left[C_1\frac{\sin\left(\sqrt{\lambda_1}LX\right)}{\sqrt{\lambda_1}} - C_2\frac{\cos\left(\sqrt{\lambda_1}LX\right)}{\sqrt{\lambda_1}} - \frac{\lambda_2}{\lambda_1}LX\right] + \frac{C_4}{(T_s - T_c)} - \frac{T_c}{(T_s - T_c)} \tag{17}$$

Results

The given values in the table below for thephysiological and physical parameters has used for acquiring the results for temperature profile in the region under research work.

K	0.499 J/m sec K
C_p	3799 J/Kg K
T_c	310 K
T_s	302 K
T_a	278 K
L	1
g	$3.28595*10^{-3.}$

Discussion

The temperature distribution in tissue and veins is presented in Figs 2-5. Fig. 2 is a plot obtained between tissue temperature (θ_1) and distance from the core (x) for different values of coefficient of metabolism. The value of metabolic heat generation and perfusion rate is taken as constant. It is observed the temperature distribution profile is parabolic, with the increase in

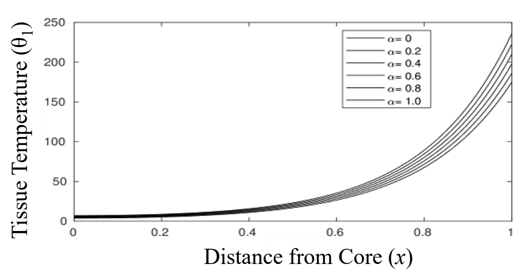

Fig. 2: Variation of tissue temperature with respect to distance from core for different values of coefficient of metabolism.

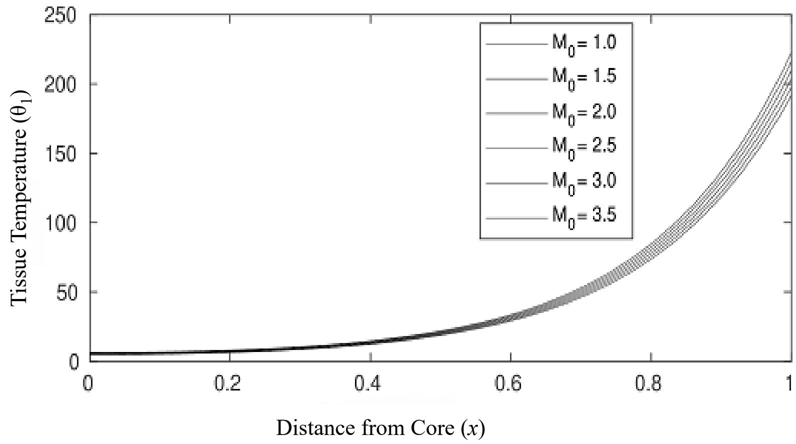

Fig. 3: Variation of tissue temperature with respect to distance from core for different values of metabolic heat generation

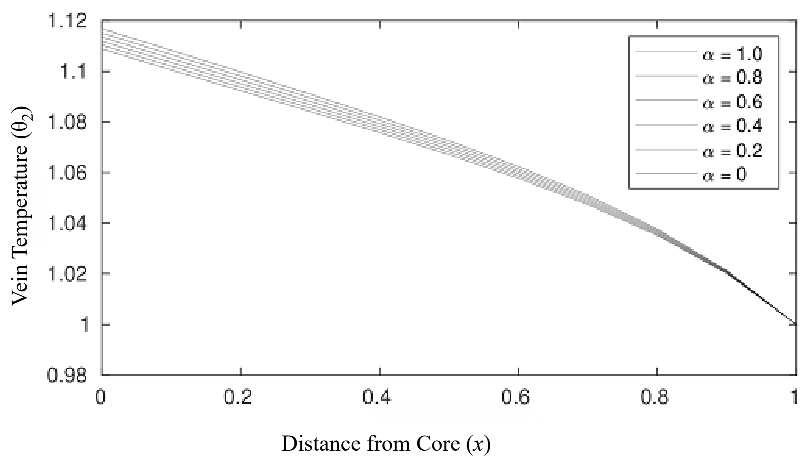

Fig. 4: Variation of vein temperature with respect to distance from core for different values of coefficient of metabolism.

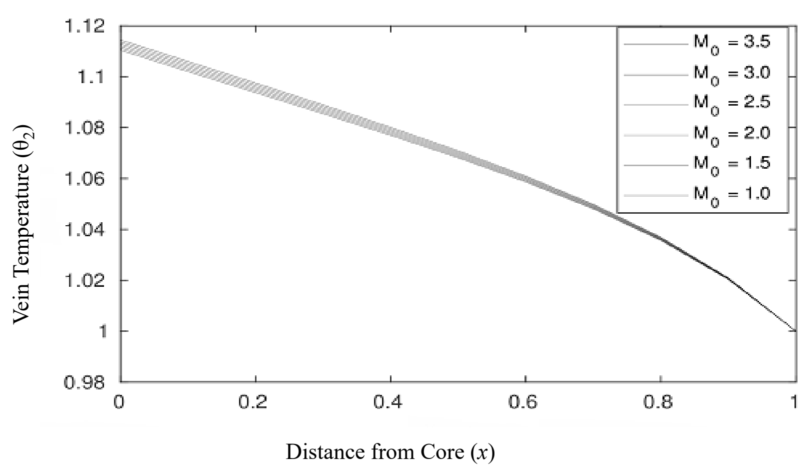

Fig. 5: Variation of vein temperature with respect to distance from core for different values of metabolic heat generation

temperature as the distance from the core increases. The metabolic factor or coefficient of metabolism increases the heat in the tissue. Fig. 3 shows the relation between tissue temperature (θ_1)and distance from the core (x) for different values of metabolic heat generation. For obtaining this curve, the perfusion rate and coefficient of metabolism are kept at a constant value. It is observed the temperature distribution profile is the same, i.e., parabolic, with the increase in temperature as the distance from the core increases. So, it is found that metabolic heat generation helps to increase heat near the surface of the skin. Fig. 4 is a plot obtained between vein temperature (θ_2) and distance from the core (x) for different values of coefficient of metabolism. The value of metabolic heat generation and perfusion rate is taken as constant. It is observed that the temperature distribution profile to be decreasing as the distance from the core increases. The values of vein temperature are significantly less as compared to tissue temperature. So this would not create a significant deviation from tissue temperature. Fig. 5 presents the relation between vein temperature (θ_2) and distance from the core (x) for different values of metabolic heat generation. It is observed that the temperature distribution profile decreases as the distance from the core increases. This plot also confirms the same behaviour of vein temperature as above.

Conclusion

The results obtained through mathematical modelling of the problem, it can be concluded that overall, the heat in tissue increases from core to surface in both cases where the coefficient of metabolism (α) and metabolic heat generation (M_0) are varied. This trend is the opposite in the case of venous blood temperature. In veins, overall, the temperature decreases from core to surface by varying the same factors. And artery temperature remains constant throughout. The model presented in this paper can be extended further to study the temperature distribution in tissue in the presence of a specific disease, and thus the model can be helpful for the development of a mechanism to cure or control the disease. In addition to this, modern mathematical techniques like fractional derivative methods viz. Hilfer fractional equation or Langevin fractional equation method could be employed. The stability analysis could be obtained through these methods.

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